Learning Goal: I will be able to determine the second derivative of a function AND connect and apply derivatives to motion.

Minds On: What do you notice

Action: Note and examples

Consolidation: Exit Question

New Unit

Applications of Derivatives

MINDS ON

The derivative of the derivative function is called the second derivative. If the first derivative is y = f'(x), we write the second derivative as f''(x).

Find the first and second derivatives with respect to x of the equation given below.

$$f(x) = 4x^3 - 3x^2 - x + 5$$

$f'(x) = 12x^2 - 6x - 1$

$f''(x) = 24x - 6$

MINDS ON

Find the first and second derivatives with respect to *t* of the equation given below.
Assume v_i is a constant.

$$d = v_i t + \frac{1}{2}at^2$$ (position) m

$d' = v_i + at$ (velocity) m/s

the rate at which distance is changing

$d'' = a$ (acceleration) m/s^2

the rate at which velocity is changing

- When we take the first derivative of the position function, s(t), it represents the velocity of the object at time t. $v(t) = s'(t) = \frac{ds}{dt}$
- When we take the second derivative of the position function, s(t), it represents the acceleration of the object at time t. $a(t) = v'(t) = s''(t)$, or $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
- Negative velocity indicates that an object is moving in a negative direction at time t.
- Negative acceleration indicates that the velocity is decreasing at time t.
- An object is accelerating (speeding up) when its velocity and acceleration have the same signs.
- An object is decelerating (slowing down) when its velocity and acceleration have opposite signs.
- The speed of an object is the magnitude of its velocity at time t.

ACTION

Example 1: Determine the second derivative of

$f(x) = \frac{x}{1+x}$ when x = 1.

$$f(x) = (x)(1+x)^{-1}$$

$$f'(x) = (1)(1+x)^{-1} + (x)(-1)(1+x)^{-2}(1)$$

$$= \frac{1}{(1+x)}\frac{(1+x)}{(1+x)} - \frac{x}{(1+x)^2}$$

$$= \frac{1+x-x}{(1+x)^2}$$

$$= \frac{1}{(1+x)^2} = (1+x)^{-2}$$

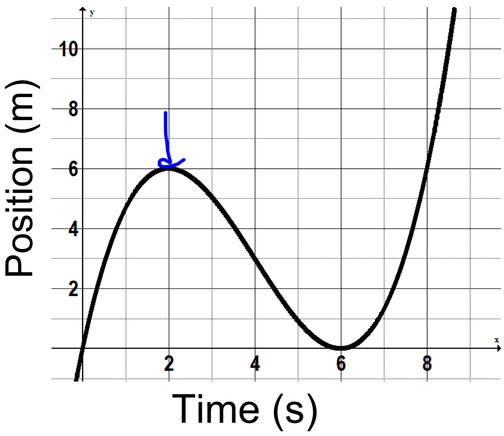
$$f''(x) = (-2)(1+x)^{-3}$$

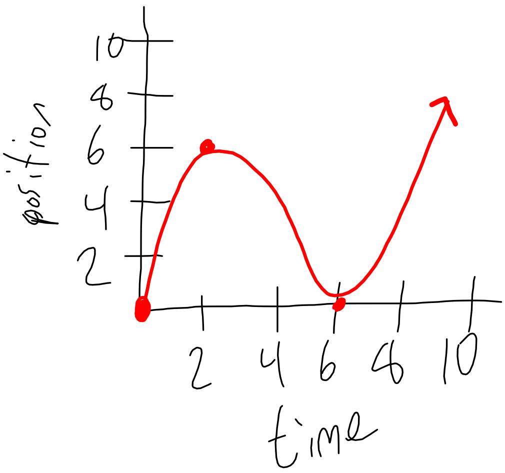
$$= \frac{-2}{(1+x)^3}$$

$$f''(1) = \frac{-2}{(1+1)^3}$$

$$= \frac{-2}{8} = -\frac{1}{4}$$

Example 2: An object is moving along a straight line. Its position, s(t), to the right of a fixed point is given by the graph shown. When is the object moving to the right, when is it moving to the left, and when is it at rest?





Moving Right
$0 \leq t < 2, t > 6$

At Rest
$t = 2, 6$

Moving Left
$2 < t < 6$

Example 3: The position of an object moving on a line is given by $s(t) = 6t^2 - t^3$, $t \geq 0$, where s is in metres and t is in seconds.

a) Determine the velocity and acceleration of the object at t = 2.

b) At what time(s) is the object at rest?

c) In which direction is the object moving at t = 5?

d) When is the object moving in a positive direction?

e) When does the object return to its initial position?

$s(t) = 6t^2 - t^3$

$v(t) = s'(t) = 12t - 3t^2$

$a(t) = v'(t) = s''(t) = 12 - 6t$

a) $v(2) = 12 m/s$ $a(2) = 0 m/s$

b) $v(t) = 0$

$12t - 3t^2 = 0$

$-3t(t-4) = 0$

$t = 0, 4 \text{ s}$

c) $v(5) = 12(5) - 3(5)^2$

$= -15 \text{ m/s}$

moving to the left

d) $v(t) > 0$

$12t - 3t^2 > 0$

$-3t(t-4) > 0$

look at $t = 0, 4$

when $t = 1$, $v(1) = 9 \text{ m/s}$ +

when $t = 5$, $v(5) = -15 \text{ m/s}$

∴ moving in a positive direction when

$0 < t < 4$

e) $s(t)=0$

$6t^2-t^3=0$

$t^2(6-t)=0$

$t=0, t=6$

Example 4: Discuss the motion of an object moving on a horizontal line if its position is given by $s(t) = t^2 - 10t$, $0 \le t \le 12$, where s is in metres and t is in second. Include the initial velocity, final velocity, and any acceleration in your discussion.

Initial Position

$s(0) = 0m$

Final Position

$s(12) = 24m$

Initial Velocity

$v(t) = 2t - 10$

$v(0) = -10 m/s$

Final Velocity

$v(12) = 14 m/s$

Moving Left

$2t - 10 < 0$

$t < 5$

$0 \le t < 5$

At Rest

$2t - 10 = 0$

$t = 5$

Moving Right

$2t - 10 > 0$

$t > 5$

$5 < t \le 12$

Acceleration

$a(t) = 2 m/s^2$

Example 5: A baseball is hit vertically upward. The position function s(t), in metres of the ball above the ground is $s(t) = -5t^2 + 30t + 1$, where t is in seconds.

a) Determine the maximum height reached by the ball.

b) Determine the velocity of the ball when it is caught 1 m above the ground.

a) at max height $v(t)=0$

$v(t) = -10t + 30$

$-10t + 30 = 0$

$t = 3s$

max height is $s(3) = -5(3)^2 + 30(3) + 1$

$= 46m$

b) We want $v(t)$ when $s(t)=1$

$\cancel{1} = -5t^2 + 30t \cancel{+1}$

$-5t^2 + 30t = 0$

$-5t(t-6) = 0$

$\cancel{t=0}$ or $t=6$

when ball is hit

$v(6) = -10(6) + 30$

$= -30\text{ m/s}$

CONSOLIDATION

A flaming arrow is shot into the air to mark the beginning of a circus act. The height, in metres, of the flaming arrow after t seconds is represented by the function
$h(t) = -4.905t^2 + 28.5t + 2$.

a) Determine the velocity and acceleration of the arrow after 4 seconds.

$v(t) = -9.81t + 28.5$
$v(4) = -10.74 m/s$

$a(t) = -9.81$
$a(4) = -9.81 m/s^2$

b) When does the arrow reach its maximum height? What is the maximum height?

$v(t) = 0$

$0 = -9.81t + 28.5$
$t = 2.9 s$

$s(2.9) = 43.4 m$

c) How long does it take for the arrow to hit the ground? At what velocity does it hit the ground?

c) How long does it take for the arrow to hit the ground?
At what velocity does it hit the ground?

$$0 = -4.905t^2 + 28.5t + 2$$

$$t = \frac{-28.5 \pm \sqrt{(28.5)^2 - 4(-4.905)(2)}}{2(-4.905)}$$

$$t = 5.9\,s$$

v(5.9) = -29.4 m/s