Minds On

Consider the function f(x) as shown below. What do you think happens for very large positive numbers of x? What about for very large negative numbers of x? Can you use this to describe the end behaviour of the function?

$$f(1000) = \frac{2998}{1005} = 2.98$$

$$f(x) = \frac{3x - 2}{x + 5}$$

$$f(-1000) = \frac{-3002}{-995} = 3.02$$

as $x \rightarrow +\infty$ $f(x) \rightarrow 3$

as $x \rightarrow -\infty$ $f(x) \rightarrow 3$

$$f(x) = \frac{3x \cancel{- 2}}{x \cancel{+ 5}}$$

Action

5.3 Graphs of Rational Functions Continued

Example 1

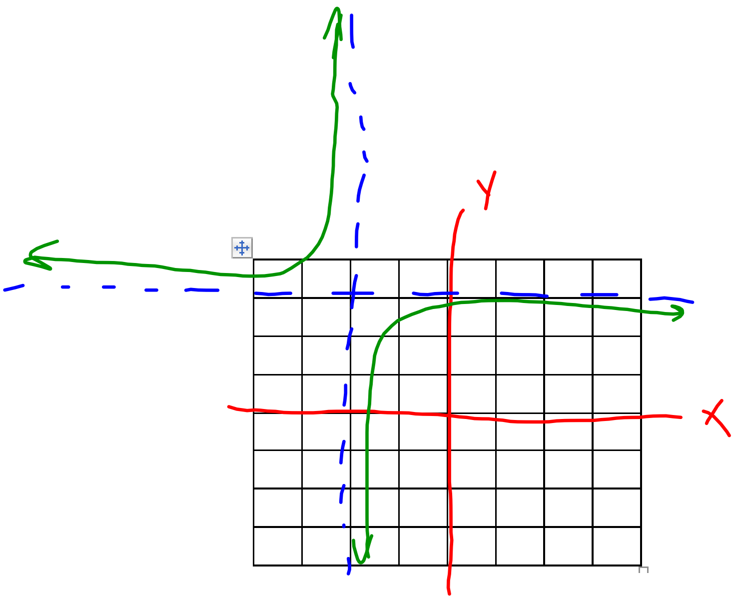
Determine how the graph of $f(x) = \frac{3x-5}{x+2}$ approaches its vertical asymptote.

right side

$$f(1.99) = \frac{3(-1.99)-5}{(-1.99)+2} = -1097$$

left side

$$f(-2.01) = \frac{3(-2.01)-5}{(-2.01)+2} = 1103$$



As $x \to -2$ from the right, $f(x) \to -\infty$

As $x \to -2$ from the left, $f(x) \to +\infty$

Action

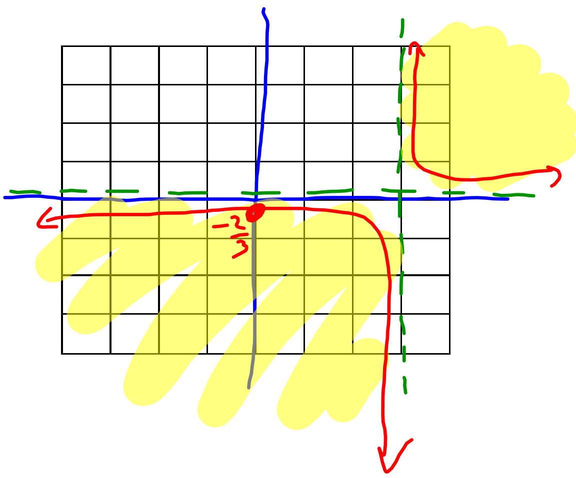
Example 2

For each function:
i) determine the domain, intercepts, asymptotes, and positive/negative intervals
ii) use these characteristics to sketch the graph of the function
iii) describe where the function is increasing or decreasing

a) $f(x) = \frac{2}{x-3}$

$x - 3 \neq 0$
$x \neq 3$

Domain $= \{x \in \mathbb{R} \mid x \neq 3\}$



Intercepts: y-intercept set $x = 0$
y-intercept $= -\frac{2}{3}$
x-intercept set $y = 0$
$0 = \frac{2}{x-3}$ no x-intercept
$0 = 2$

Asymptotes vertical: $x = 3$
horizontal: $y = 0$

Positive intervals $f(x) > 0$ when $x - 3 > 0$
$x > 3$

Negative intervals $f(x) < 0$ when $x - 3 < 0$
$x < 3$

The function is decreasing when $(-\infty, 3), (3, +\infty)$

Action

b) $f(x) = \dfrac{x-2}{3x+4}$

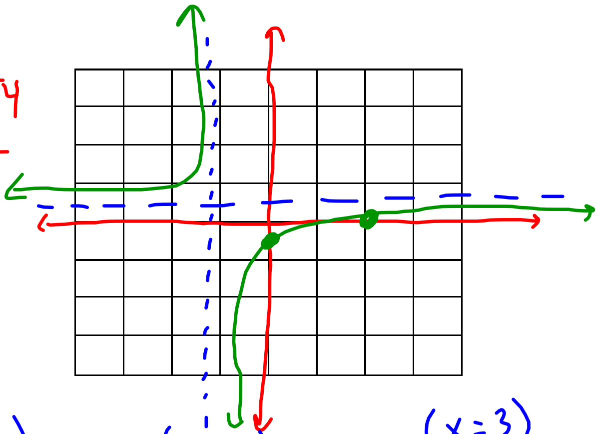
Domain $= \{x \in \mathbb{R} \mid x \neq -\frac{4}{3}\}$

Intercepts y-intercept $= -\frac{1}{2}$

x-intercept: $0 = \dfrac{x-2}{3x+4}$

$0 = x-2$

$x = 2$



Asymptotes

vertical: $x = -\frac{4}{3}$

horizontal: $y = \frac{1}{3}$

	$(x=-2)$ $x < -\frac{4}{3}$	$(x=0)$ $-\frac{4}{3} < x < 2$	$(x=3)$ $x > 2$
numerator $(x-2)$	$-$	$-$	$+$
denominator $(3x+4)$	$-$	$+$	$+$
function	$+$	$-$	$+$

Action

c) $f(x) = \dfrac{x-3}{2x-6}$ $\quad 2x-6 \neq 0$

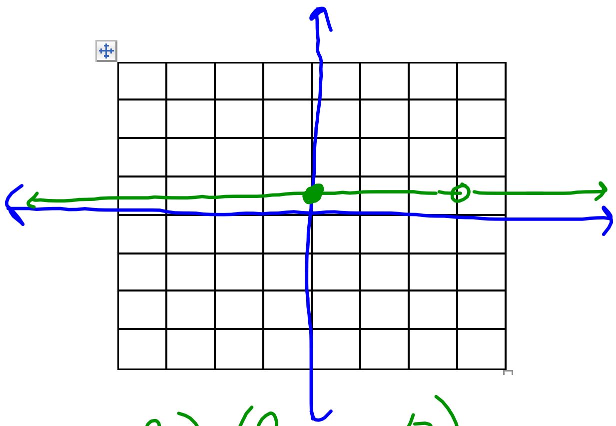
Domain $= \{x \in \mathbb{R} \mid x \neq 3\}$

y-intercept: $y = \frac{1}{2}$

x-intercept: $\cancel{x=3}$
no x-intercept

$f(x) = \dfrac{\cancel{(x-3)}}{2\cancel{(x-3)}}$

$f(x) = \frac{1}{2}$



$(-\infty, 3), (3, +\infty)$

f(x) is always positive

Sketch a graph of the function below using as much information as you can.

$$f(x) = \frac{4x + 8}{2x - 4}$$

$$f(x) = \frac{4(x+2)}{2(x-2)}$$

Domain: $\{x \in \mathbb{R} \mid x \neq 2\}$

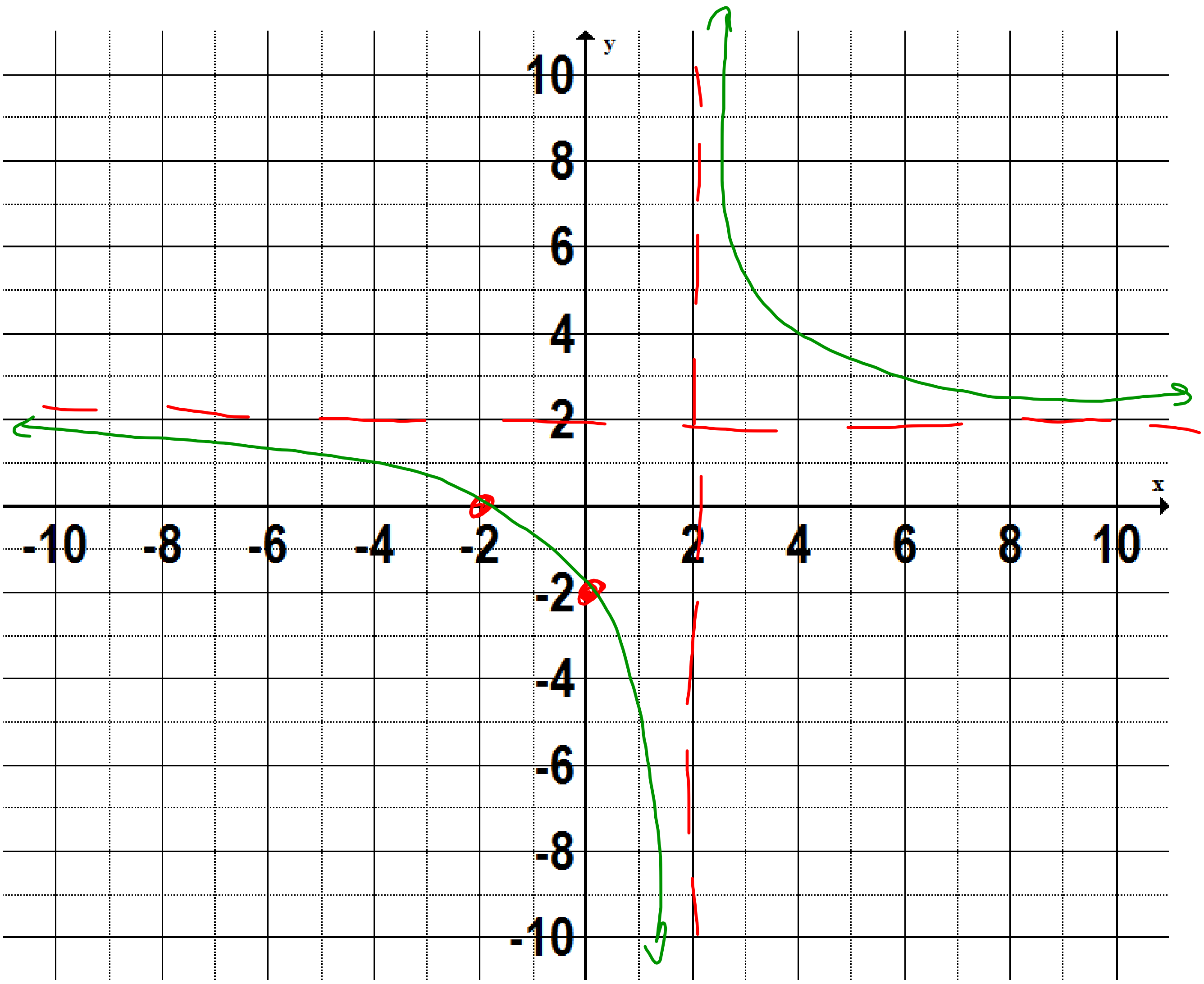
y-intercept = -2

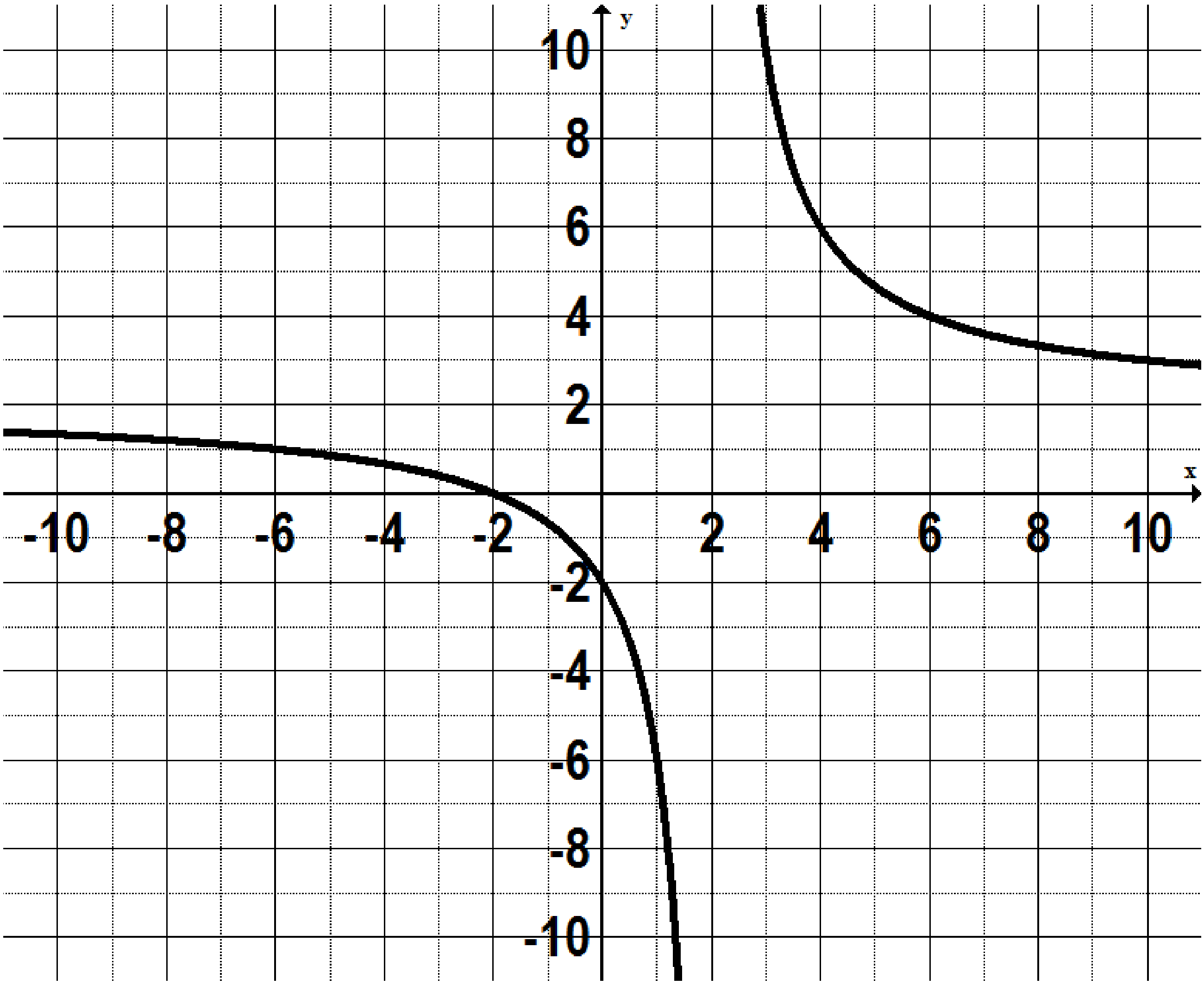
x-intercept = -2

horizontal asymptote: $y = 2$

vertical asymptote: $x = 2$

	$(x=-3)$ $x < -2$	$(x=0)$ $-2 < x < 2$	$(x=3)$ $x > 2$
$4x+8$	$-$	$+$	$+$
$2x-4$	$-$	$-$	$+$
$\frac{4x+8}{2x-4}$	$+$	$-$	$+$


y
10
8
6
4
2
-2
-4
-6
-8
-10
x
-10
-8
-6
-4
-2
2
4
6
8
10


y
x
10
8
6
4
2
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
2
4
6
8
10

Action

Key Ideas

The graphs of most rational functions of the form $f(x) = \dfrac{b}{cx + d}$ and $f(x) = \dfrac{ax + b}{cx + d}$ have both a vertical asymptote and a horizontal asymptote.

You can determine the equation of the vertical asymptote directly from the equation of the function by finding the zero of the denominator.

You can determine the equation of the horizontal asymptote directly from the equation of the function by examining the ratio f the leading coefficients in the numerator and the denominator. T his give you the end behaviours of the function.

To sketch the graph of a rational function, you can use the domain, intercepts, equations of asymptotes, and positive/negative intervals.

Action

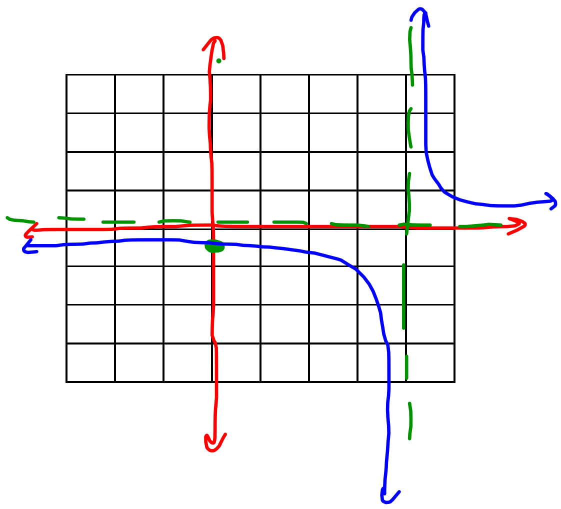
Need to Know

Rational functions of the form $f(x) = \dfrac{b}{cx + d}$ have a vertical asymptote defined by x = -d/c and a horizontal asymptote defined by y = 0.

For Example:

* no x-intercept

$$f(x) = \frac{2}{x-4}$$

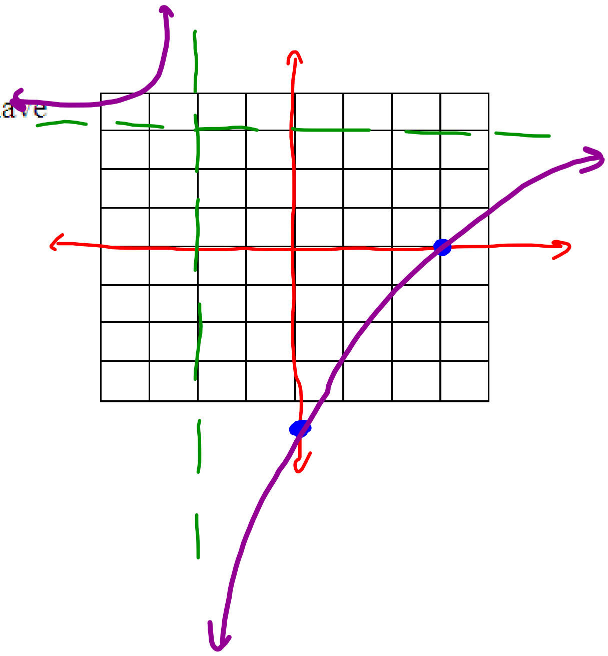


vertical asymptote: $x = 4$

y-intercept: $y = -0.5$

Action

Most rational functions of the form $f(x) = \frac{ax+b}{cx+d}$ have a vertical asymptote defined by x = -d/c and a horizontal asymptote defined by y = a/c.
For Example:



$$f(x) = \frac{3x-9}{x+2}$$

x-intercept: $3x - 9 = 0$
$x = 3$

y-intercept: $y = -\frac{9}{2}$ $y = -4.5$

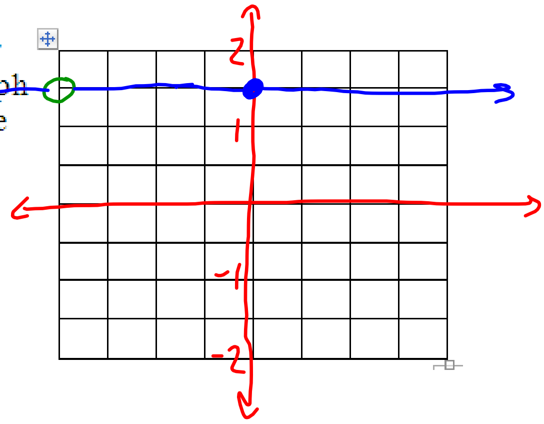
horizontal asymptote: $y = 3$

vertical asymptote: $x = -2$

Action

HOWEVER, when the numerator and the denominator both contain a common linear factor, the result is a graph of a horizontal line that has a hole where the zero of the common factor occurs. Thus, the graph has no asymptotes.

For Example:



$$f(x) = \frac{3x+6}{2x+4}$$

$$= \frac{3\cancel{(x+2)}}{2\cancel{(x+2)}}$$

$$= \frac{3}{2} \qquad x \neq -2$$

Consolidation

State the equation of a rational function that meets the conditions below:

vertical asymptote at **x = -1**

horizontal asymptote at **y = 3/2**

positive when **x < -1** and when **x > 2**

negative when **-1 < x < 2**

$3x - 6 = 2$

$$f(x) = \frac{3x - 6}{2x + 2}$$

Consolidation

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