Learning Goal: I will be able to interpret the interactions between 3 planes.

Minds On: What are the possible options?

Action: Note: 9.4

Consolidation: Planning

9.4 The Intersection of Three Planes

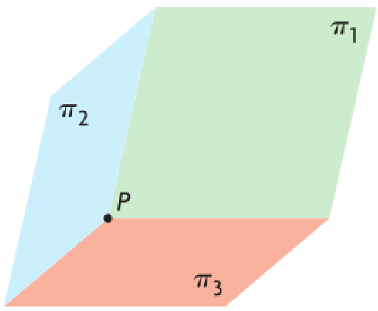
Minds On: In what ways might 3 planes interact? Use the table to classify your resonponses as Consistent Systems or Inconsistent Systems.

Consistent Systems	Inconsistent Systems
All 3 coincident	Triangular Prism Creates 3 // lines
2 Coincident 1 cuts through	All // non-coincident
3 Planes meet @ a point	2 //, 1 cuts through both of those
All 3 meet in a line	2 coincident, 1 parallel

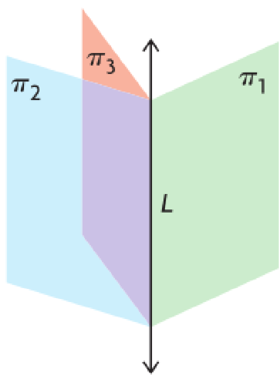
Intersection of 3 Planes

Consistent Systems

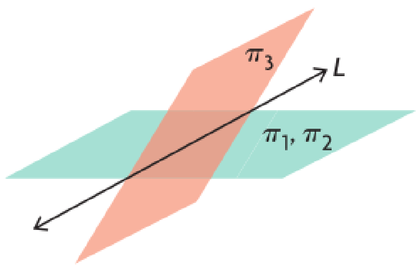
Case 1: P is the point of intersection of three planes.



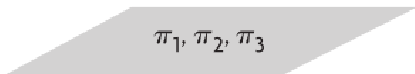
Case 2a: L is the line of intersection of three planes.



Case 2b: L is the line of intersection of two coincident planes and a third plane not parallel to the coincident planes.

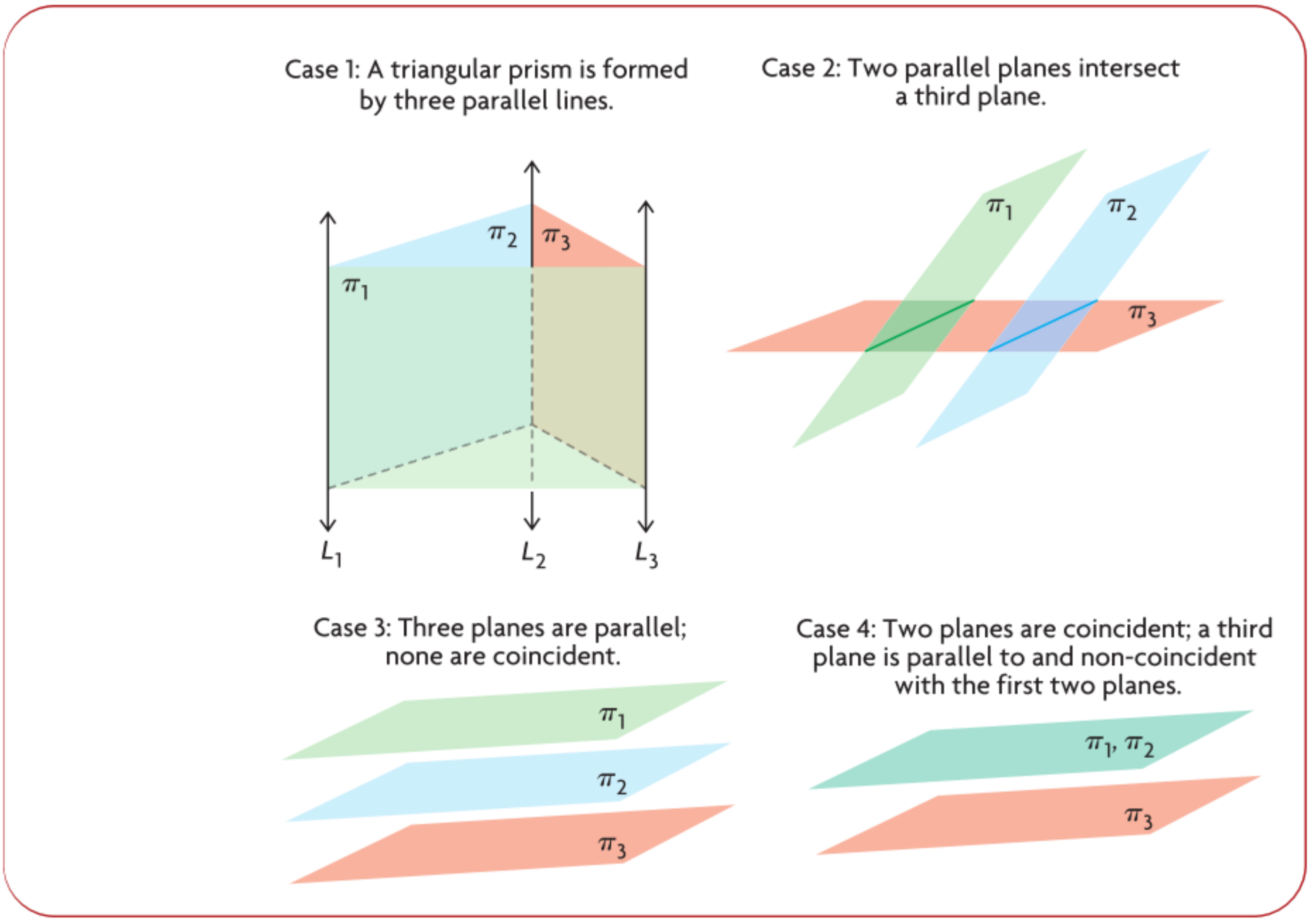


Case 3: The plane of intersection of three coincident planes is the plane itself.



Intersection of 3 Planes

Inconsistent Systems



Example 1: Determine the intersection of the three planes with the equations

$x - y + z = -2$, ① $2x - y - 2z = -9$, ② and $3x + y - z = -2$ ③

$$-2x + 2y - 2z = 4 \quad ① \times -2$$
$$+\ 2x - y - 2z = -9 \quad ②$$
$$0x + y - 4z = -5 \quad ④$$

$$-3x + 3y - 3z = 6 \quad ① \times -3$$
$$3x + y - z = -2 \quad ③$$
$$0x + 4y - 4z = 4 \quad ⑤$$

$$0x - 4y + 16z = 20 \quad ④ \times -4$$
$$+\ 0x + 4y - 4z = 4 \quad ⑤$$
$$0x + 0y + 12z = 24 \quad ⑥$$

$$0x + 0y + z = 2 \quad ⑥ \times \frac{1}{12}$$

$$\therefore \boxed{z = 2}$$

Sub $z=2$ into ④

$0x + y - 4(2) = -5$

$\boxed{y=3}$

Sub $y=3, z=2$ into ①

$x-(3)+(2)=-2$

$x=-2-2+3$

$\boxed{x=-1}$

Check $(x,y,z)=(-1,3,2)$ in ② + ③

Example 2: Determine the solution to the following system of equations:

1) $2x - y + z = 1$ ①
2) $3x - 5y + 4z = 3$ ②
3) $3x + 2y - z = 0$ ③

$\vec{n}_1 = (2, -1, 1)$
$\vec{n}_2 = (3, -5, 4)$
$\vec{n}_3 = (3, 2, -1)$

No parallel or coincident planes.

1. Eliminate x from ② + ③

$$-6x + 3y - 3z = -3 \quad ① \times -3$$
$$6x - 10y + 8z = 6 \quad ② \times 2$$
$$0x - 7y + 5z = 3 \quad ④$$

$$-6x + 3y - 3z = -3 \quad ① \times -3$$
$$6x + 4y - 2z = 0 \quad ③ \times 2$$
$$0x + 7y - 5z = -3 \quad ⑤$$

2. Eliminate y using ④ + ⑤

$$-7y + 5z = 3 \quad ④$$
$$7y - 5z = -3 \quad ⑤$$

$$\boxed{0y + 0z = 0}$$

* consistent

$\therefore$ infinite solutions... Case 2a

~~meet at a line.~~

solution is a line.

Let $\boxed{z = t}$ in (4)

$$-7y + 5(t) = 3$$

$$-7y = 3 - 5t$$

$$\boxed{y = \frac{-3}{7} + \frac{5}{7}t} \quad (6)$$

Sub $z = t$, $\underbrace{y = \frac{-3}{7} + \frac{5}{7}t}_{(6)}$ into (1)

$$2x - \left(-\frac{3}{7} + \frac{5}{7}t\right) + (t) = 1$$

$$2x + \frac{3}{7} - \frac{5}{7}t + t = 1$$

$$2x = 1 - \frac{3}{7} + \frac{5}{7}t - t$$

$$2x = \frac{4}{7} - \frac{2}{7}t$$

$$x = \frac{4}{14} - \frac{2}{14}t$$

$$\boxed{x = \frac{2}{7} - \frac{1}{7}t}$$

$\therefore (x, y, z) = \left(\frac{2}{7} - \frac{1}{7}t, -\frac{3}{7} + \frac{5}{7}t, t\right)$

or $x = \frac{2}{7} - 1t$

$y = -\frac{3}{7} + 5t$

$z = 7t$

solution is $\vec{r} = \left(\frac{2}{7}, -\frac{3}{7}, 0\right) + t(-1, 5, 7)$

(X) Check (x, y, z)
in (2) & (3)

** Book also checked in (1)

Example 3: Determine the solution the following system of equations:

1. $2x + y + z = 1$ $\quad \vec{n}_1 = (2,1,1)$ $\quad D_1 = 1$
2. $4x - y - z = 5$ $\quad \vec{n}_2 = (4,-1,-1)$ $\quad D_2 = 5$
3. $8x - 2y - 2z = 10$ $\quad \vec{n}_3 = (8,-2,-2)$ $\quad D_3 = 10$

③ = 2×② $\therefore$ ② and ③ are coincident.

$\therefore$ ① is $\not\parallel$ to ② or ③, we have a solution.

The 3 planes will meet along a line

$$-4x - 2y - 2z = -2 \quad (1) \times -2$$
$$+ \quad 4x - y - z = 5 \quad (2)$$
$$0x - 3y - 3z = 3 \quad (4)$$

$$-4x - 4y - 4z = -4 \quad (1) \times -4$$
$$+ \quad 4x - 2y - 2z = 10 \quad (3)$$
$$0x - 6y - 6z = 6 \quad (5)$$

$0x + 6y + 6z = -6$ (4) $\times -2$

$0x - 6y - 6z = 6$ (5)

$0x + 0y + 0z = 0$

As expected, infinite solutions. (line)

Set $\boxed{z = t}$ in (4)

$0x - 3y - 3(t) = 3$

$3y = -3 - 3t$

$\boxed{y = -1 - t}$

Sub $y = -1-t$ and $z = t$ into ①

$$2x + (-1-t) + (t) = 1$$

$$2x - 1 - \cancel{t} + \cancel{t} = 1$$

$$\frac{2x = 2}{\boxed{x = 1}}$$

$\therefore (x, y, z) = (1, -1-t, t)$

or $\vec{r} = (1, -1, 0) + t(0, -1, 1)$

Example 4: Determine the solution the following systems of equations:

1. $x - y + z = 1$ ①
2. $x + y + 2z = 2$ ②
3. $x - 5y - z = 1$ ③

Either - meet @ point
- meet along line
- inconsistent triangular prism

No ∥ planes

$$-x + y - z = -1 \quad ① \times -1$$
$$+ \quad x + y + 2z = 2 \quad ②$$
$$0x + 2y + z = 1 \quad ④$$

$$-x + y - z = -1 \quad ① \times -1$$
$$+ \quad x - 5y - z = 1 \quad ③$$
$$0x - 4y - 2z = 0 \quad ⑤$$

Eliminate y using ④ + ⑤

$$0x + 4y + 2z = 2 \quad (4) \times 2$$
$$+ \; 0x - 4y - 2z = 0 \quad (5)$$
$$\overline{\qquad 0y + 0z = 2 \qquad}$$

No solution!

$\therefore$ triangular prism situation

Now, show that the lines where each pair of planes meet are parallel.

$\vec{n}_1 \times \vec{n}_2 = (1,-1,1) \times (1,1,2)$

$$\begin{array}{cc} -1 & 1 \\ 1 & 2 \\ 1 & 1 \\ -1 & 1 \end{array}$$

$x = (-2-1)$
$x = -3$
$y = (1-2)$
$= -1$
$z = (1-(-1))$
$z = 2$

$\boxed{= (-3,-1,2)}$

$\vec{n}_1 \times \vec{n}_3 = (1,-1,1) \times (1,-5,-1)$

$$\begin{array}{cc} -1 & -5 \\ 1 & -1 \\ 1 & 1 \\ -1 & -5 \end{array}$$

$x = 6$
$y = 2$
$z = -4$

$\boxed{= (6,2,-4)}$

$x = -2(\vec{n}_1 \times \vec{n}_2)$

$\therefore$ parallel

$$\vec{n}_2 \times \vec{n}_3 = (1,1,2) \times (1,-5,-1)$$

$$\begin{array}{cc} 1 & -5 \\ 2 & -1 \\ 1 & 1 \\ 1 & -5 \end{array} \quad \begin{array}{l} x = 9 \\ y = 3 \\ z = -6 \end{array}$$

$$= (9,3,-6)$$

$$= -3(\vec{n}_1 \times \vec{n}_2)$$

$\therefore$ the 3 L.O.I. are ∥

Example 5: Determine the solution the following systems of equations:

1. $x - y + 2z = -1$ ①
2. $x - y + 2z = 3$ ②
3. $x - 3y + z = 0$ ③

$$\boxed{\begin{aligned}\tilde{n}_1 &= (1, -1, 2)\\ \tilde{n}_2 &= (1, -1, 2)\end{aligned}} \quad \begin{aligned}D_1 &= -1\\ D_2 &= 3\end{aligned}$$

① ∥ ② and non-coincident.

∴ no solution!

Example 6: Solve the following system of equations:

1. $x + y + z = 5$ $\quad \vec{n}_1 = (1,1,1) \quad D_1 = 5$
2. $x + y + z = 4$ $\quad \vec{n}_2 = (1,1,1) \quad D_2 = 4$
3. $x + y + z = 5$ $\quad \vec{n}_3 = (1,1,1) \quad D_3 = 5$

$① = ③$ and $② \neq k(① + ③)$

We have 2 coincident planes (① & ③) and a third plane (②) that is // and non-coincident with the other two.

$\therefore$ no solution!