Learning Goal: I will be able to use the second derivative to determine features of a graph.

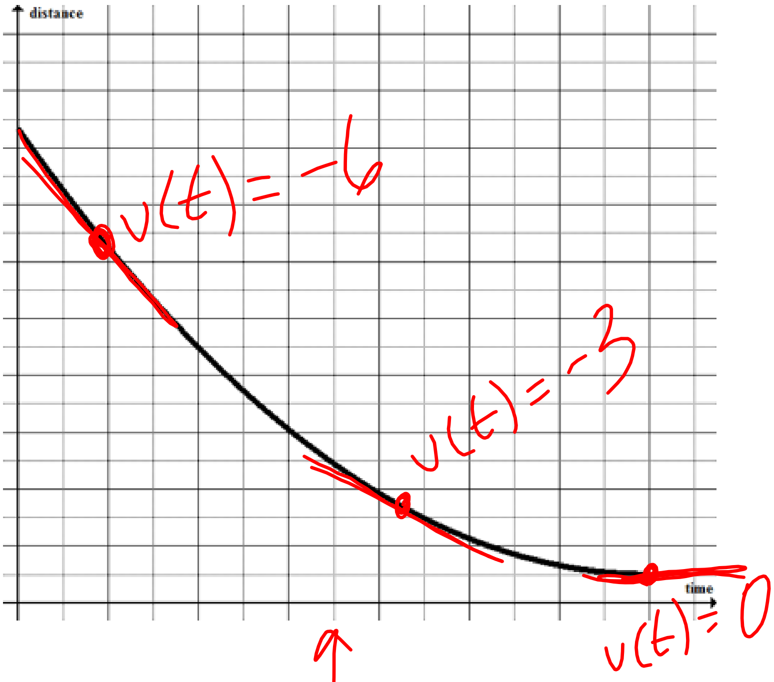
Minds On: Distance-Time Graphs

Action: Concavity and Points of Inflection

Consolidation: Exit Question

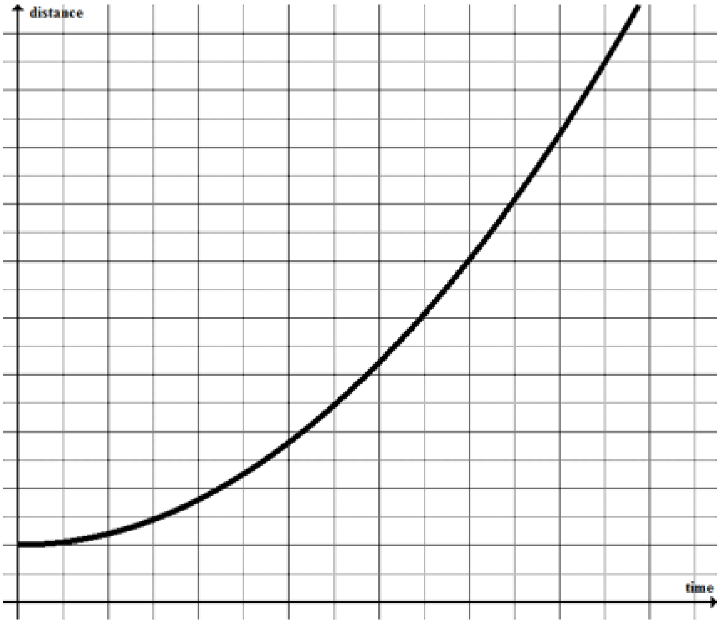
Minds On Distance-Time Graphs

The graph of a function f(x) is **concave up** on an interval if f'(x) is increasing on the interval. The graph of a function f(x) is **concave down** on an interval if f'(x) is decreasing on the interval.



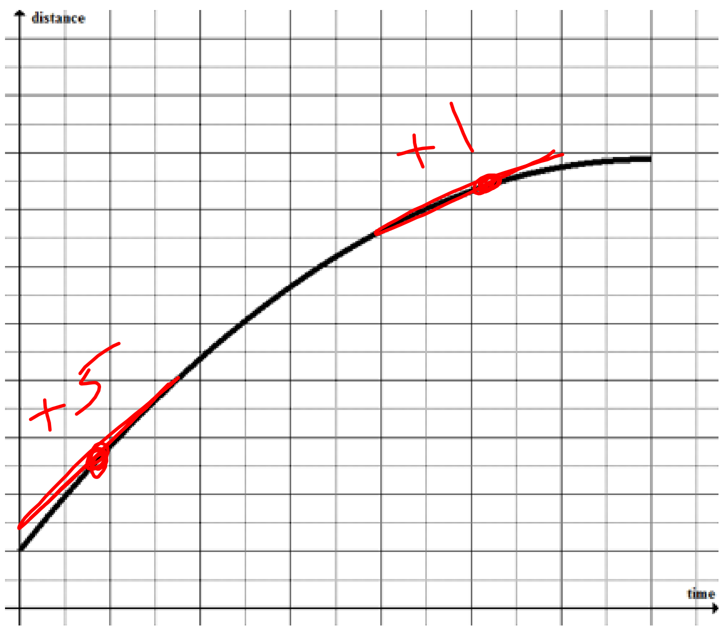
Velocity is ↑ ______________

Acceleration is (+) ______________



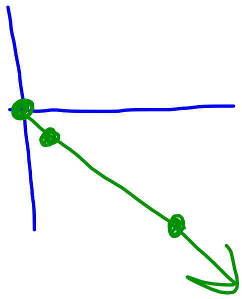
Velocity is ↑

Acceleration is +



Velocity is ____________________

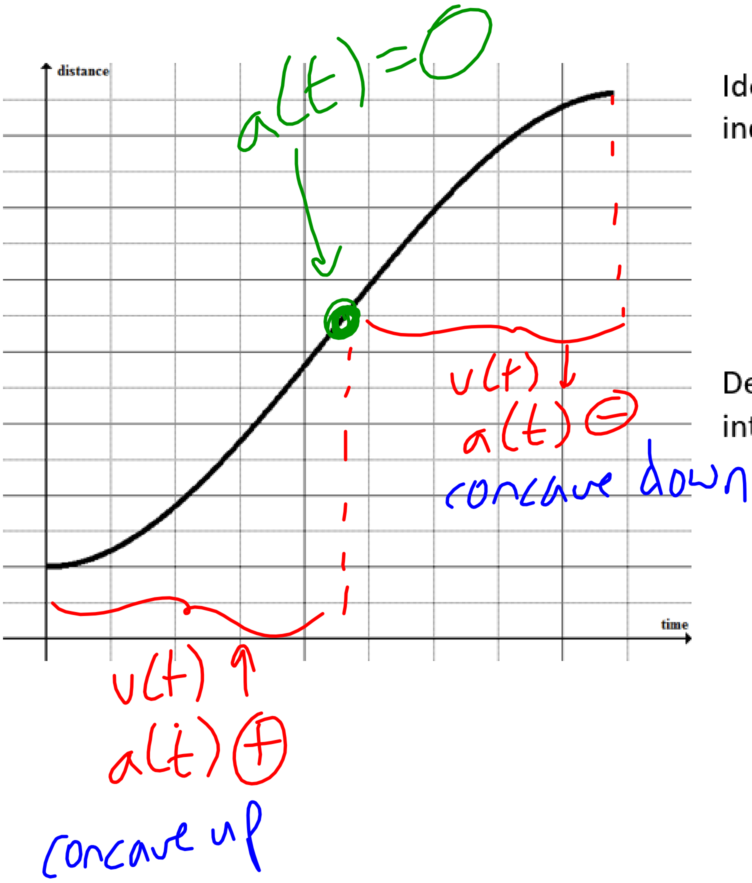
Acceleration is ____ − ____



Velocity is $\downarrow$

Acceleration is $-$

A **point of inflection** is a point on the graph of f(x) where the function changes from concave up to concave down, or vice versa.



Identify intervals over which the velocity is increasing and decreasing.

Describe the acceleration over each interval.

Action

Concavity and Points of Inflection

Test for Concavity: If f(x) is a differentiable function whose second derivative exists on an open interval I, then

- The graph of f(x) is concave up on I if f''(x) > 0 for all values of x in I
- The graph of f(x) is concave down on I if f''(x) < 0 for all values of x in I

Test for Point of Inflection: If f(x) is a differentiable function whose second derivative exists on an interval containing c then

- The graph of f(x) has a point of inflection where the function changes from concave up to concave down, or vice versa.
- $f''(c) = 0$ or is undefined if (c, f(c)) is a point of inflection on the graph of f(x).

<u>The second derivative test:</u> Suppose that f(x) is a function for which f'(c) = 0, and the second derivative of f(x) exists on an interval containing c.

critical numbers

- If f''(c) > 0, then f(c) is a local minimum value.
- If f''(c) < 0, then f(c) is a local maximum value.
- If f''(c) = 0, then the test fails. Use the first derivative test.

Example 1: Sketch the graph of $y = x^3 - 3x^2 - 9x + 10$ no holes/asymptotes

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

$\frac{dy}{dx} = 0$ when $3x^2 - 6x - 9 = 0$
$3(x^2 - 2x - 3) = 0$
$3(x-3)(x+1) = 0$

critical numbers:
$x = 3, -1$

$$\frac{d^2y}{dx^2} = 6x - 6$$

$\frac{d^2y}{dx^2} = 0$ when $6x - 6 = 0$
$x = 1$

	$x<1$	$x=1$	$x>1$
$f''(x)$	$-$	0	$+$
graph	concave down	point of inflection	concave up
sketch			

Critical Numbers: $x = 3, -1$

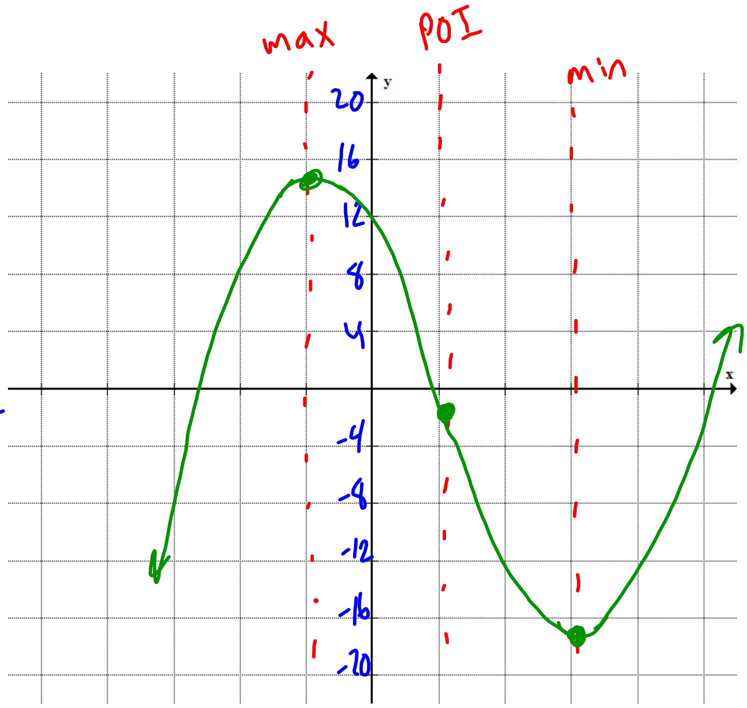
When $x = -1$, $\frac{d^2y}{dx^2}$ is negative

$\therefore$ max

When $x = 3$, $\frac{d^2y}{dx^2}$ is positive

$\therefore$ min

$f(-1) = 15$
$f(1) = -1$
$f(3) = -17$



Example 2: Sketch the graph of $f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$f'(x) = 0$ when $x = 0$

$f''(x) = 0$ when $x = 0$ maybe POI?

look at $f''(x)$ when $x < 0$ and $x > 0$

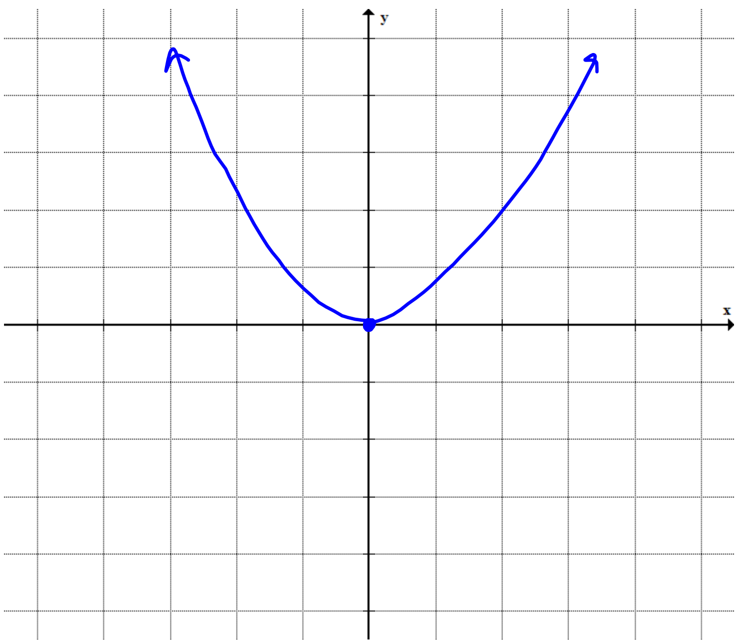
	$x<0$	$x=0$	$x>0$
$f''(x)$	+	0 fail	+
graph	concave up	? not POI	concave up
sketch	∪	?	∪

First derivative test

when $x<0$, $f'(x)$ ⊖

when $x>0$, $f'(x)$ ⊕

∴ minimum



Example 3: Sketch the graph of the function f(x) = $x^{1/3}$ $\sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$= \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$

$f'(x) = 0$ when never

$f'(x)$ DNE when $x = 0$ *critical number

$f''(x) = \frac{-2}{9}x^{-5/3}$

$= \frac{-2}{9x^{5/3}}$

$f''(x) = 0$ when never

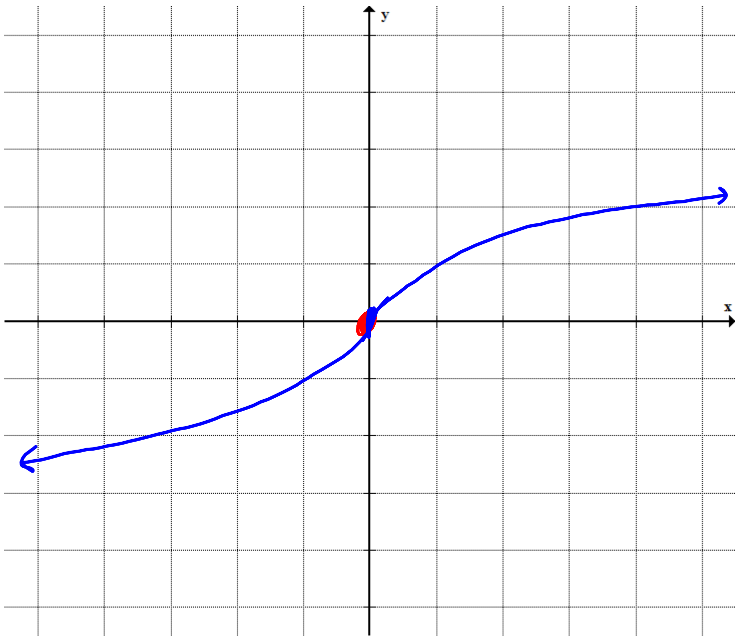
$f''(x)$ DNE when $x = 0$

	$x<0$	$x=0$	$x>0$
$f''(x)$	$\frac{-}{-} = +$	DNE	$\frac{-}{+} = -$
graph	concave up	POI	concave down
sketch	∪		∩

as $x \to 0$ what happens to $f'(x)$?

$f'(x) \to +\infty$

vertical tangent



Example 4: Determine any points of inflection on the graph of $f(x) = \frac{1}{x^2+3}$

↳ happens when $f''(x) = 0$ or DNE <u>and</u> graph goes from concave up to concave down or vice versa.

Find $f''(x)$

$f(x) = (x^2+3)^{-1}$

$f'(x) = -1(x^2+3)^{-2}(2x)$

$f'(x) = \frac{-2x}{(x^2+3)^2} = -2x(x^2+3)^{-2}$

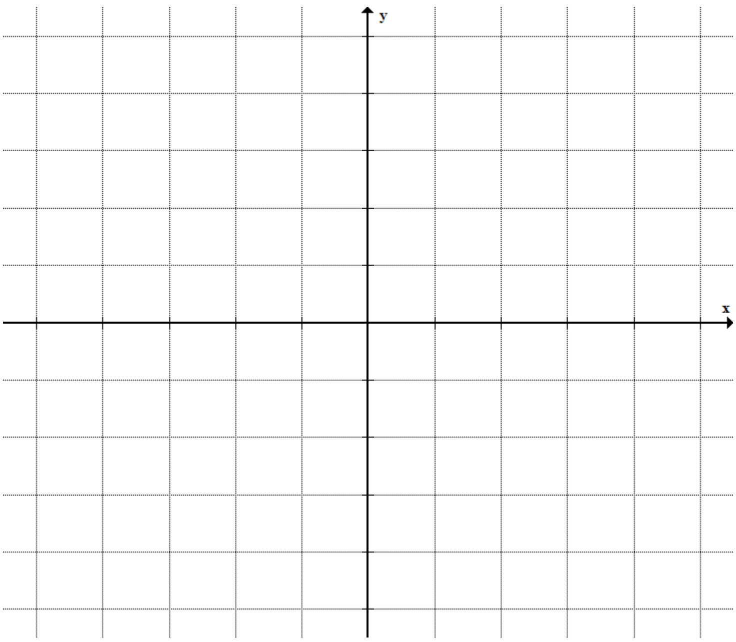
** Pg 205 #1, 5

$f''(x) = -2(x^2+3)^{-2} + -2x(-2)(x^2+3)^{-3}(2x)$

$= \frac{-2(x^2+3)}{(x^2+3)^2(x^2+3)} + \frac{8x^2}{(x^2+3)^3}$

$= \frac{-2x^2-6+8x^2}{(x^2+3)^3}$

$= \frac{6x^2-6}{(x^2+3)^3}$



Consolidation

A Proper Sketch

What elements should be included in a "good" graph?