Learning Goal: I will be able to find derivatives using implicit differentiation.

Minds On: Take the derivatives of...

Action: Implicit Differentiation

Consolidation: Practice and Exit Question

Minds On

Find dy/dx for each of the following.

1) $y = 2x^2 - 3x - 2$

$$\frac{dy}{dx} = 4x - 3$$

2) $3x + 4y = 8$

$$y = 2 - \frac{3}{4}x$$

$$\frac{dy}{dx} = -\frac{3}{4}$$

3) $x^2 + y^2 = 9$

$$y = \sqrt{9-x^2} = (9-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(9-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-2x}{2}(9-x^2)^{-\frac{1}{2}}$$

$$= \frac{-x}{\sqrt{9-x^2}}$$

4) $4xy + 3y^2 = 12$

$$y(4x+3y) = 12$$

$$y = \frac{12}{4x+3y}$$

$$y = 12(4x+3y)^{-1}$$

$$=$$

$$\frown$$

Action

Implicit Differentiation

We're used to functions written as $\boxed{y =}$ or $\boxed{f(x) =}$, where y is isolated on one side and is expressed explicitly in terms of our x variable. Many functions are defined implicitly, such as the circle $x^2 + y^2 = 16$.

Since there are x-values that correspond to two y-values, y is not a function of x on the entire circle. (fails the vertical line test)

To solve for y we would get y = $\pm\sqrt{16-x^2}$

where $+\sqrt{16-x^2}$ represents the upper semicircle and $-\sqrt{16-x^2}$ the lower.

If we were asked to determine the slope of the tangent to the circle at the point (0,4), we would differentiate the function $+\sqrt{16-x^2}$ and sub in x = 0. This process required us to first rearrange the equation for y, then choose which part of the equation we were going to use, then differentiate the equation, then solve.

In this circle problem, we could have prepared to take the derivative, y', by first rearranging for y. There are many situations where solving for y in terms of x is very difficult, and sometimes impossible. In these cases, we use an alternative method called <u>implicit differentiation</u>.

To fully understand implicit differentiation, we need to remember/highlight a couple of things:

1. $\frac{dy}{dx}$ vs $\frac{d}{dx}$

derivative of y w.r.t. x

derivative of something w.r.t. x

2. How the chain rule comes into play: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Action

Example 1: a) If $x^2 + y^2 = 25$, determine dy/dx. b) Determine the slope of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, -4).

To find $\frac{dy}{dx}$, take the derivative of each part w.r.t. x

a) $$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 0$$

$$2x + 2y \times \frac{dy}{dx} = 0$$

$$\frac{2y\frac{dy}{dx}}{2y} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

b) slope when $(x,y) = (3,-4)$ $= \frac{-3}{-4} = \frac{3}{4}$

$$x^2+y^2=25$$

Take the derivative old school.

$$y=\pm\sqrt{25-x^2}$$ rewrite rearranged for y

separate into two parts

$$y=+\sqrt{25-x^2}$$

$$y=(25-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{\cancel{2}}(25-x^2)^{-\frac{1}{2}}(-\cancel{2}x)$$

$$=\frac{-x}{\sqrt{25-x^2}}$$

$$=\frac{-x}{y}$$

$$y=-\sqrt{25-x^2}$$

$$y=-(25-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{-1}{\cancel{2}}(25-x^2)^{-\frac{1}{2}}(-\cancel{2}x)$$

$$=\frac{x}{-\sqrt{25-x^2}}$$

$$=\frac{-x}{y}$$

Find dy/dx **without** first rearranging for y.

2) 3x + 4y = 8

$$\frac{d}{dx}(3x+4y)=\frac{d}{dx}(8)$$

$$\frac{d}{dx}3x+\frac{d}{dx}4y=\frac{d}{dx}8$$

$$3+\frac{d(4y)}{dy}\times\frac{dy}{dx}\ \text{show}=0$$

$$3+4\times\frac{dy}{dx}=0$$

$$\frac{\cancel{4}\times\frac{dy}{dx}}{\cancel{4}}=-\frac{3}{4}$$

$$\frac{dy}{dx}=-\frac{3}{4}$$

3) $x^2 + y^2 = 9$

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(9)$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(9)$$

$$2x+\frac{d(y^2)}{dy}\times\frac{dy}{dx}=0$$

$$2x+2y\cdot\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-x}{y}$$

4) $4xy + 3y^2 = 12$

$$\frac{d}{dx}(4xy+3y^2) = \frac{d}{dx}(12)$$

$$\frac{d}{dx}(4xy) + \frac{d(3y^2)}{dx} = \frac{d}{dx}(12)$$

product rule!

* derivative of first part × second
+
first part × derivative of second

$$\left(\frac{d(4x)}{dx}\right)y + 4x\left(\frac{d}{dx}(y)\right) + \frac{d(3y^2)}{dy} \times \frac{dy}{dx} = 0$$

$$4y + 4x\left(\frac{d(y)}{dy} \times \frac{dy}{dx}\right) + 6y \times \frac{dy}{dx} = 0$$

unneccessarryy because $\frac{d(y)}{dx}$ is $\frac{dy}{dx}$

$$4y + 4x\left(1 \times \frac{dy}{dx}\right) + 6y \times \frac{dy}{dx} = 0$$

$$4y + 4x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(4x+6y) = -4y$$

$$\frac{dy}{dx} = \frac{-4y}{4x+6y}$$

Action

Example 2: Determine dy/dx for $2xy - y^3 = 4$.

$$\frac{d}{dx}(2xy - y^3) = \frac{d}{dx}(4)$$

$$\underbrace{\frac{d}{dx}(2xy)}_{\text{Product Rule}} - \frac{d}{dx}(y^3) = \frac{d}{dx}(4)$$

$$\left(\frac{d}{dx}(2x)\right)y + 2x\left(\frac{d}{dx}(y)\right) - \frac{d(y^3)}{dy} \times \frac{dy}{dx} = 0$$

$$2y + 2x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2y}{2x - 3y^2}$$

Action

Procedure for Implicit Differentiation

If an equation defines y implicitly as a differentiable function of x, determine dy/dx as follows:

1. Differentiate both side of the equation with respect to x. Remember to use the chain rule when differentiating terms containing y.
2. Solve for dy/dx.

Consolidation

Exit Question

Determine dy/dx for **(x + y)³ = 12x**

$$\frac{d}{dx}(x+y)^3 = \frac{d}{dx}(12x)$$

$$3(x+y)^2 \times \frac{d}{dx}(x+y) = 12$$

$$3(x+y)^2 \times \left(\frac{d}{dx}(x) + \frac{d}{dx}(y)\right) = 12$$

$$\frac{3(x+y)^2\left(1 + \frac{dy}{dx}\right)}{3(x+y)^2} = \frac{12}{3(x+y)^2}$$

$$1 + \frac{dy}{dx} = \frac{4}{(x+y)^2}$$

$$\frac{dy}{dx} = \frac{4}{(x+y)^2} - 1$$