Learning Goal: I will be able to determine the scalar and vector projections of a vector.

Minds On: What do we know so far?

Action: Scalar and Vector Projections

Consolidation: Check-in yesterday's exit card
Exit question - Page 399 #6

Minds On: Dot Product Recap

1. Make up two vectors in R^3
2. Determine the dot product.
3. Swap vectors with a classmate.
4. Calculate each other's dot products.
5. Compare.

Do you agree??

$$(-3, 1, 2) \cdot (4, -3, 1)$$

$$= (-3)(4) + (1)(-3) + (2)(1)$$

$$= -12 - 3 + 2$$

$$= -13$$

Action

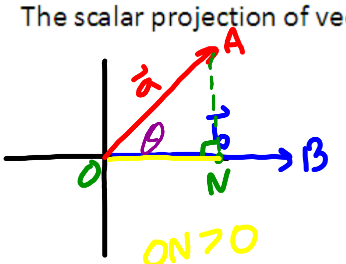
7.5 Scalar and Vector Projections

When two vectors, $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ are placed tail to tail, and θ is the angle between the vectors, $0^\circ \le \theta \le 180^\circ$, the scalar projection of $\vec{a}$ on $\vec{b}$ is ON, as shown below. The scalar projection can be determined using right triangle trigonometry and can be applied to either algebraic or geometric vectors equally well.

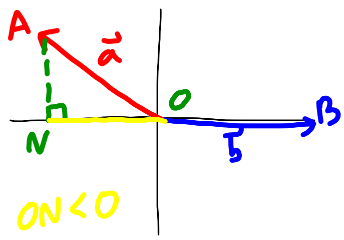
The scalar projection of $\vec{a}$ on $\vec{b}$ is obtained by drawing a line from the head of vector $\vec{a}$ perpendicular to $\overrightarrow{OB}$, or an extension of $\overrightarrow{OB}$.

Scalar Projection of $\vec{a}$ on $\vec{b}$

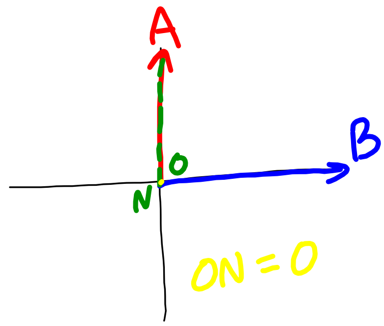
The scalar projection of vector $\vec{a}$ onto $\vec{b}$ is ON, where ON = $|\vec{a}|cos\theta$.



$0 \le \theta < 90^\circ$



$90^\circ < \theta \le 180^\circ$

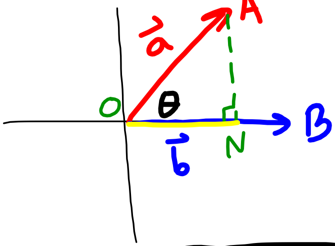


$\theta = 90^\circ$

Action

The scalar projection of vector $\vec{a}$ on vector $\vec{b}$ is in general not equal to the scalar projection of vector $\vec{b}$ on vector $\vec{a}$, which can be seen as follows:

Calculating the scalar projection of $\vec{a}$ on $\vec{b}$:



$\cos\theta = \frac{ON}{|\vec{a}|}$

$\boxed{ON = |\vec{a}|\cos\theta}$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\vec{a}\cdot\vec{b} = |\vec{b}|\,ON$

$ON = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Calculating the scalar projection of $\vec{b}$ on $\vec{a}$:

$$ON = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$\overbrace{|\vec{a}|\cos\theta}^{ON} = \overbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}}^{ON}$$

Action

Calculating Scalar Projections

The scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. The scalar projection of $\vec{b}$ on $\vec{a}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$. In general, $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \neq \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

Action

Example 1: a) Show algebraically that the scalar projection of $\vec{a}$ on $\vec{b}$ is identical to the scalar projection of $\vec{a}$ on $2\vec{b}$.

b) Show algebraically that the scalar projection of $\vec{a}$ on $2\vec{b}$ is not the same as $\vec{a}$ on $-2\vec{b}$.

a) $$|\vec{a}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$|\vec{a}|\cos\theta = \frac{\vec{a}\cdot 2\vec{b}}{|2\vec{b}|}$$

$$= \frac{\vec{a}\cdot \cancel{2}\vec{b}}{\cancel{2}|\vec{b}|}$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$\vec{a}$ on $2\vec{b}$

$$ON = \frac{\vec{a} \cdot 2\vec{b}}{|2\vec{b}|} = \frac{\vec{a} \cdot \cancel{2}\vec{b}}{\cancel{2}|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$\vec{a}$ on $-2\vec{b}$

$$ON = \frac{\vec{a} \cdot -2\vec{b}}{|-2\vec{b}|} = \frac{\vec{a} \cdot -\cancel{2}\vec{b}}{\cancel{2}|\vec{b}|} = \frac{\vec{a} \cdot -\vec{b}}{|\vec{b}|} = \frac{-(\vec{a} \cdot \vec{b})}{|\vec{b}|}$$

$\neq$

Action

Example 2: For the vectors $\vec{a}$ = (-3, 4, 5√3) and $\vec{b}$ = (-2, 2, -1), calculate scalar projections of $\vec{a}$ on $\vec{b}$.

$$ON = |\vec{a}| \cos\theta$$

$$ON = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$ON = \frac{(-3, 4, 5\sqrt{3}) \cdot (-2, 2, -1)}{\sqrt{(-2)^2 + (2)^2 + (-1)^2}}$$

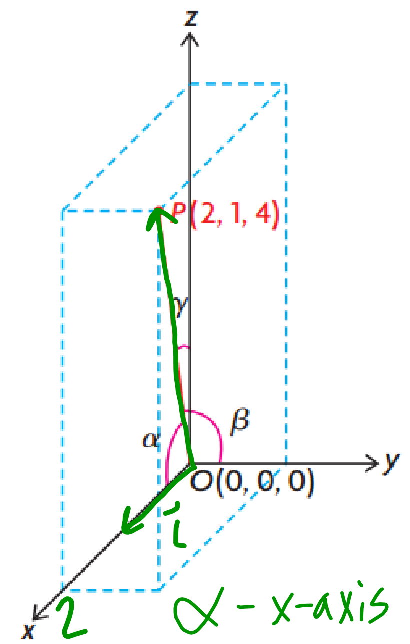
$$= \frac{(6) + (8) + (-5\sqrt{3})}{\sqrt{9}}$$

$$= \frac{14 - 5\sqrt{3}}{3}$$

$$= 1.79$$

Action

Example 3: Determine the angle that the vector $\overrightarrow{OP}$ = (2, 1, 4) makes with each of the co



α - x-axis
β - y-axis
γ - z-axis

$$\overrightarrow{OP} \cdot \vec{i} = |\overrightarrow{OP}||\vec{i}|\cos\alpha$$

$$\cos\alpha = \frac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}||\vec{i}|}$$

$$= \frac{(2,1,4)\cdot(1,0,0)}{\left(\sqrt{2^2+1^2+4^2}\right)(1)}$$

$$= \frac{(2)(1)+(1)(0)+(4)(0)}{\sqrt{21}}$$

$$\cos\alpha = \frac{2}{\sqrt{21}}$$

$$\alpha \doteq 64.1^\circ$$

$$\cos\beta = \frac{\vec{OP} \cdot \vec{j}}{|\vec{OP}||\vec{j}|}$$

$$= \frac{(2,1,4)\cdot(0,1,0)}{(\sqrt{21})(1)}$$

$$\cos\beta = \frac{1}{\sqrt{21}}$$

$$\beta \doteq 77.4^\circ$$

$$\gamma = \cos^{-1}\left(\frac{\vec{OP} \cdot \vec{k}}{|\vec{OP}||\vec{k}|}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{21}}\right)$$

$$\doteq 29.2^\circ$$

Action

$(a, b, c) \cdot (1, 0, 0)$

Direction Cosines for $\overrightarrow{OP}$ = (a, b, c)

The angles α, β, and γ are the angles that vector $\overrightarrow{OP}$ = (a, b, c) makes the positive x-axis, y-axis and z-axis respectively.

To determine these angles:

$$\cos\alpha = \frac{a}{|\overrightarrow{OP}|} = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

$$\beta = \cos^{-1}\left(\frac{b}{|\overrightarrow{OP}|}\right)$$

$$\gamma = \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

all equivalent definition forms.

Action

Example 4: For the vector $\overrightarrow{OP}$ = (-2√2, 4, -5), determine the direction cosine and corresponding angle that this vector makes with the positive z-axis.

$\cos\alpha$ α

$$\cos\alpha = \left(\frac{-2\sqrt{2}}{\sqrt{(-2\sqrt{2})^2 + (4)^2 + (-5)^2}}\right)$$

$(-2\sqrt{2})(-2\sqrt{2})$
$=(-2)(-2)(\sqrt{2})(\sqrt{2})$
$=(4)(2)$
$=8$

$$= \frac{-2\sqrt{2}}{\sqrt{(8)+16+25}}$$

$$= \frac{-2\sqrt{2}}{\sqrt{49}}$$

$$\boxed{\cos\alpha = \frac{-2\sqrt{2}}{7}}$$

$$\alpha = \cos^{-1}\left(\frac{-2\sqrt{2}}{7}\right)$$

$$\alpha \doteq 113.8^\circ$$

$$\cos\beta = \frac{4}{7}$$

$$\beta \doteq 55.2^\circ$$

$$\cos\gamma = \frac{-5}{7} \qquad \gamma \doteq 135.6^\circ$$

Action

Examining Vector Projections

The calculation of the vector projection of $\vec{a}$ on $\vec{b}$ is just the corresponding scalar projection of $\vec{a}$ on $\vec{b}$ multiplied by $\frac{\vec{b}}{|\vec{b}|}$. The expression $\frac{\vec{b}}{|\vec{b}|}$ is a unit vector pointing in the direction of $\vec{b}$.

$\frac{1}{|\vec{b}|}\vec{b}$

Vector Projection of $\vec{a}$ on b

Vector projection of $\vec{a}$ on $\vec{b}$ = (scalar projection of $\vec{a}$ on $\vec{b}$)(unit vector in the direction of $\vec{b}$)

$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$ $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$ $= \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}, \vec{b} \neq \vec{0}$

ON or scalar projection

vector projection = (scalar projection)(unit vector // to $\vec{b}$)

Action

Example 5: Find the vector projection of $\overrightarrow{OA} = (4,3)$ on $\overrightarrow{OB} = (4, -1)$.

1. Find scalar projection.

$$ON = \frac{(4,3)\cdot(4,-1)}{\sqrt{(4)^2+(-1)^2}}$$

$$= \frac{(16)+(-3)}{\sqrt{17}}$$

$$= \frac{13}{\sqrt{17}}$$

Now, we want a vector that is the same length as ON, but that is parallel to $\vec{b}$.

Therefore, we need to multiply our scalar projection, ON, by a vector of length 1 that is in line with $\vec{b}$.

length of $\vec{b} = \sqrt{17}$

Multiply by $\frac{1}{\sqrt{17}}$

$$\left(\frac{1}{\sqrt{17}}\sqrt{17}\right) = 1$$

$\therefore$ vector projection =

$$\left(\frac{13}{\sqrt{17}}\right)\underbrace{\left(\frac{1}{\sqrt{17}}(4,-1)\right)}_{\text{unit vector in same direction as } \vec{b}}$$

$$= \frac{13}{17}(4,-1)$$

$$= \left(\frac{52}{17}, \frac{-13}{17}\right)$$

Consolidation:

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