Learning Goal: I will be able to perform a cross product on two algebraic vectors in 3D.

Minds On: Example 1 - you try!

Action: Note and practice on page 407

Consolidation: Exit Question and
Introduction to unit assessment

Minds On

Finishing Yesterday

✓

See yesterday's
lesson file

Action

7.6 The Cross Product of Two Vectors

The cross product is often referred to as a vector product, because, when it is calculated, the result is a vector and not a scalar. If we are given two vectors, $\vec{a}$ and $\vec{b}$, and wish to calculate their cross product, what we are trying to find is a particular vector that is perpendicular to each of the two given vectors. As you will see, there is actually an infinite number of vectors perpendicular to the two vectors in R^3.

Example 1: Given the vectors $\vec{a}$ = (1, 1, 0) and $\vec{b}$ = (1, 3, -1), determine $\vec{a} \times \vec{b}$.

$\vec{a} \cdot \vec{v} = 0$ and $\vec{b} \cdot \vec{v} = 0$ → $\vec{v} = (x, y, z)$

$(1,1,0) \cdot (x,y,z) = 0$

$x + y = 0$ ①

$\boxed{x = -y}$ ③

$(1,3,-1) \cdot (x,y,z) = 0$

$x + 3y - z = 0$ ②

Sub ③ into ②

$(-y) + 3y - z = 0$ ④

$2y - z = 0$

$\boxed{z = 2y}$ ⑤

$\therefore (x,y,z) = (-y, y, 2y)$

$= y(-1,1,2)$

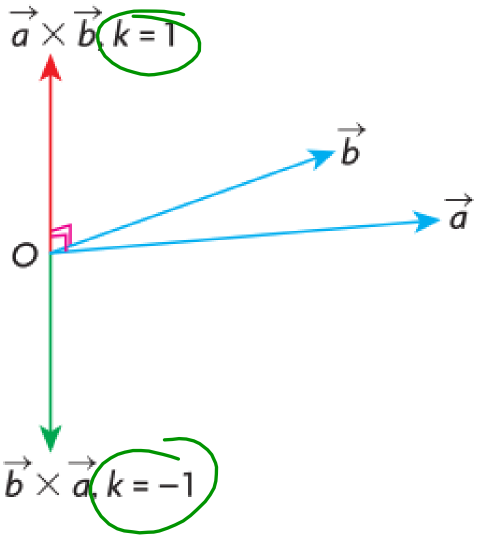
or $(x,y,z) = k(-1,1,2)$ where $k \in \mathbb{R}$

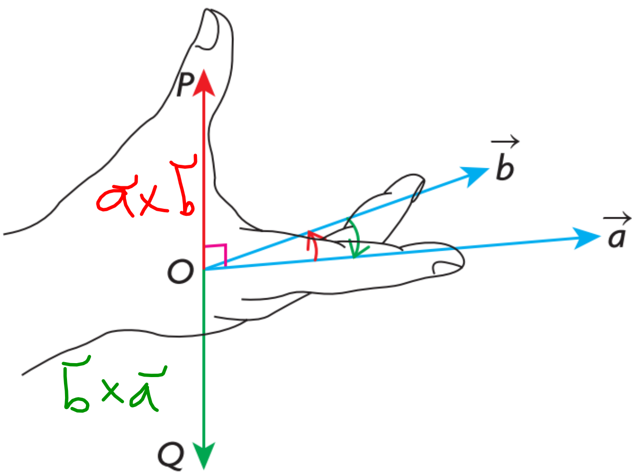
This means that the cross product of our two vectors is k(-1, 1, 2), which is a scalar multiple of (-1, 1, 2)

$\vec{a} \times \vec{b} = (-1,1,2)$ | $\vec{b} \times \vec{a} = (1,-1,-2)$

Typically, we are interested in the case where k is 1.

****READ "Deriving a Formula for the Cross Product" on page 402 before moving on.**

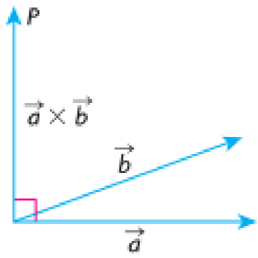

$\vec{a} \times \vec{b}$, $k = 1$
$\vec{b}$
$\vec{a}$
O
$\vec{b} \times \vec{a}$, $k = -1$


P
$\vec{a} \times \vec{b}$
$\vec{b}$
$\vec{a}$
O
$\vec{b} \times \vec{a}$
Q

Action

Definition of a Cross Product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ in R^3 is the vector that is perpendicular to these vectors such that the vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.
The vector $\vec{b} \times \vec{a}$ is the opposite of $\vec{a} \times \vec{b}$ and points in the opposite direction.



Formula for Calculating the Cross Product of Algebraic Vectors

$\boxed{k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)}$ is a vector perpendicular to both a and b, k∈**R**.

If k = 1, then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

If k = -1, then $\vec{b} \times \vec{a} = a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2)$

Action

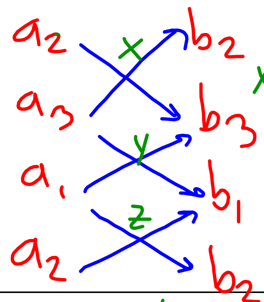
Easy to memorize, right? How about a trick to remember...

Method of Calculating $\vec{a} \times \vec{b}$, where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

1. List the components of vector $\vec{a}$ in column form on the left side, starting with a_2 and then writing a_3, a_1, and a_2 below each other as shown.

2. Write the components of vector $\vec{b}$ in a column to the right of $\vec{a}$, starting with b_2 and then writing b_3, b_1, and b_2 in exactly the same way as the components of $\vec{a}$.

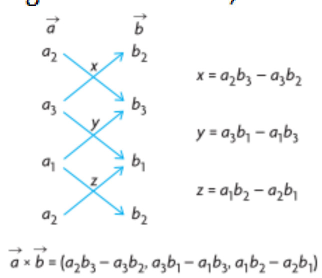
3. The required formula is now a matter of following the arrows and doing the calculation, as shown.



$x = a_2b_3 - a_3b_2$

$y = a_3b_1 - a_1b_3$

$z = a_1b_2 - a_2b_1$

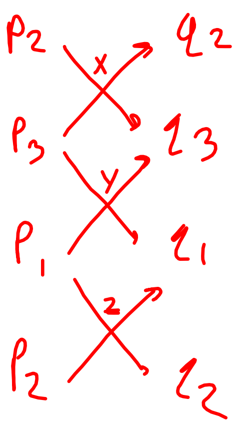


$\therefore \vec{a} \times \vec{b} = (x, y, z) = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$

$\vec{b} \times \vec{a} = (a_3b_2 - a_2b_3,\ a_1b_3 - a_3b_1,\ a_2b_1 - a_1b_2)$

Action

Example 2: If $\vec{p} = (\overset{p_1}{-1}, \overset{p_2}{3}, \overset{p_3}{2})$ and $\vec{q} = (\overset{q_1}{2}, \overset{q_2}{-5}, \overset{q_3}{6})$, calculate $\vec{p} \times \vec{q}$ and $\vec{q} \times \vec{p}$.



$$\vec{p} \times \vec{q} = (p_2q_3 - p_3q_2, p_3q_1 - p_1q_3, p_1q_2 - p_2q_1)$$

$$= ((3)(6) - (2)(-5), (2)(2) - (-1)(6), (-1)(-5) - (3)(2))$$

$$= (18 + 10, 4 + 6, 5 - 6)$$

$$\boxed{\vec{p} \times \vec{q} = (28, 10, -1)}$$

$$\vec{q} \times \vec{p} = -1(28, 10, -1)$$

$$= (-28, -10, 1)$$

Action

Properties of the Cross Product

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three vectors in R^3, and let $k \in \mathbf{R}$.

Vector multiplication is not commutative: $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

Distributive law for vector multiplication: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Scalar law for vector multiplication: $k(\vec{a} \times \vec{b}) = (k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b})$

Consolidation

Discussion/Exit Question

Draw the parallelogram determined by vectors a and b. Draw the two diagonals, a + b and a - b. Explain why (a - b) (a x b) = 0.