7.6 The Cross Product of Two Vectors

The cross product is often referred to as a vector product, because, when it is calculated, the result is a vector and not a scalar. If we are given two vectors, $\vec{a}$ and $\vec{b}$, and wish to calculate their cross product, what we are trying to find is a particular vector that is perpendicular to each of the two given vectors. As you will see, there is actually an infinite number of vectors perpendicular to the two vectors in R^3.

Example 1: Given the vectors $\vec{a}$ = (1, 1, 0) and $\vec{b}$ = (1, 3, -1), determine $\vec{a} \times \vec{b}$.

Definition of a Cross Product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ in R^3 is the vector that is perpendicular to these vectors such that the vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.
The vector $\vec{b} \times \vec{a}$ is the opposite of $\vec{a} \times \vec{b}$ and points in the opposite direction.

Formula for Calculating the Cross Product of Algebraic Vectors

$k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ is a vector perpendicular to both a and b, $k \in \mathbf{R}$.

If k = 1, then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

If k = -1, then $\vec{b} \times \vec{a} = a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2)$

Easy to memorize, right? How about a trick to remember...

Method of Calculating $\vec{a} \times \vec{b}$, where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

1. List the components of vector $\vec{a}$ in column form on the left side, starting with a_2 and then writing a_3, a_1, and a_2 below each other as shown.

2. Write the components of vector $\vec{b}$ in a column to the right of $\vec{a}$, starting with b_2 and then writing b_3, b_1, and b_2 in exactly the same way as the components of $\vec{a}$.

3. The required formula is now a matter of following the arrows and doing the calculation, as shown.

Example 2: If $\vec{p}$ = (-1, 3, 2) and $\vec{q}$ = (2, -5, 6), calculate $\vec{p} \times \vec{q}$ and $\vec{q} \times \vec{p}$.

Properties of the Cross Product

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three vectors in R^3, and let $k \in \mathbf{R}$.

Vector multiplication is not commutative: $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

Distributive law for vector multiplication: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Scalar law for vector multiplication: $k(\vec{a} \times \vec{b}) = (k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b})$