Learning Goal: I will be able to use equivalent trigonometric relationships to prove trigonometric identities.

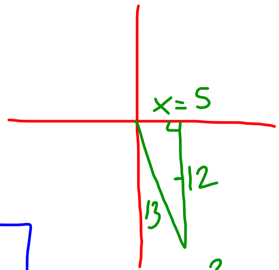
Minds On: Warmup Question

Action: This is how we prove it.

Consolidation: Additional Questions

Minds On

Determine the value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $sin\theta = -\frac{12}{13}$ and $\boxed{\frac{3\pi}{2} \leq \theta \leq 2\pi.}$



$x^2 = 13^2 - 12^2$
$x^2 = 25$
$x = 5$

We need $\sin\theta$ ✓, $\cos\theta$ ✓, $\tan\theta$ ✓

$\sin\theta = \frac{-12}{13}$

$\cos\theta = \frac{5}{13}$

$\tan\theta = \frac{-12}{5}$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{-12}{13}\right)\left(\frac{5}{13}\right)$$
$$\boxed{\sin 2\theta = \frac{-120}{169}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(\frac{5}{13}\right)^2 - \left(\frac{-12}{13}\right)^2$$
$$= \frac{25 - 144}{169}$$
$$\boxed{\cos 2\theta = \frac{-119}{169}}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$= \frac{2\left(\frac{-12}{5}\right)}{1 - \left(\frac{-12}{5}\right)^2}$$
$$= \frac{\frac{-24}{5}}{1 - \frac{144}{25}}$$
$$= \frac{\frac{-24}{5}}{\frac{25}{25} - \frac{144}{25}}$$
$$= \frac{\frac{-24}{5}}{\frac{-119}{25}}$$
$$= \frac{-24}{\cancel{5}} \times \frac{\cancel{25}\,5}{-119}$$
$$= \frac{120}{119}$$

Minds On

Proving Identities Using Other Identities

$$\cos x = \sin x \cot x$$

$$\cos(\pi - x) = -\cos x$$

$$\csc 2x = \frac{\csc x}{2\cos x}$$

$$1 - \cos^2 x = \sin x \cos x\,(\tan x - \cot x)$$

Minds On

Proving Identities Using Other Identities

$$\frac{\cos x}{\sin x} = \frac{\cancel{\sin x} \cot x}{\cancel{\sin x}}$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\cot x = \cot x \quad \checkmark$$

$$\cos x = \sin x \cot x$$

R.S.

$$= \sin x \cot x$$

$$= \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}}$$

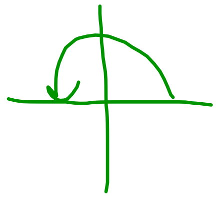
L.S. = R.S.

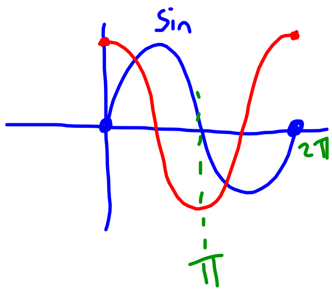
$$\therefore \cos x = \sin x \cot x$$

Minds On

Proving Identities Using Other Identities

$$\cos(\pi - x) = -\cos x$$





L.S.

$= \cos(\pi - x)$

$= \cos\pi \cos x + \sin\pi \sin x$

$= (-1)\cos x + \cancel{(0)\sin x}$

$= -\cos x$

L.S. = R.S.

$\therefore \cos(\pi - x) = -\cos x$

Minds On

Proving Identities Using Other Identities

$$\csc 2x = \frac{\csc x}{2\cos x}$$

R.S.

$= \frac{\csc x}{2\cos x}$

$= \frac{1}{2\cos x \sin x}$

$= \frac{1}{\sin 2x}$

$= \csc 2x$

L.S.

$= \csc 2x$

$= \frac{1}{\sin 2x}$

L.S. = R.S. $\therefore \csc 2x = \frac{\csc x}{2\cos x}$

Minds On

Proving Identities Using Other Identities

$$1 - 2\cos^2 x = \sin x \cos x\,(\tan x - \cot x)$$

R.S.

$= \sin x \cos x\,(\tan x - \cot x)$

$= \sin x \cos x\left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}\right)$

$= \frac{\sin x \cancel{\cos x} \sin x}{\cancel{\cos x}} - \frac{\cancel{\sin x} \cos x \cos x}{\cancel{\sin x}}$

$= \sin^2 x - \cos^2 x$

$= 1 - \cos^2 x - \cos^2 x$

$$\boxed{\begin{array}{l} * \sin^2 x + \cos^2 x = 1 \\ \sin^2 x = 1 - \cos^2 x \end{array}}$$

$= 1 - 2\cos^2 x$

L.S. = R.S.

$\therefore 1 - 2\cos^2 x = \sin x \cos x\,(\tan x - \cot x)$

Action

This is how we prove it!

We can prove trigonometric identities by:

1. Simplifying the more complicated side until it is identical to the other side.
2. Manipulating both sides to get the same expression.

While proving an identity we may be required to:

1. Rewrite expressions using our known trigonometric identities.
2. Breaking an expression into multiple parts.
3. Finding a common denominator.
4. Factoring an expression.

Example 1

Prove that $\frac{\sin 2x}{1+\cos 2x} = \tan x$.

$\underline{L.S.}$

$= \frac{\sin 2x}{1+\cos 2x}$

$= \frac{2\sin x \cos x}{\cancel{1} + 2\cos^2 x \cancel{-1}}$

$= \frac{\cancel{2}\sin x \cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x}$

$= \frac{\sin x}{\cos x}$

$= \tan x$

$L.S. = R.S.$

$\therefore \frac{\sin 2x}{1+\cos 2x} = \tan x$

Example 2

Prove that $\sin x + \sin 2x = \sin 3x$ is not an identity.

Test $x = \pi$

L.S.

$\sin \pi + \sin 2\pi$

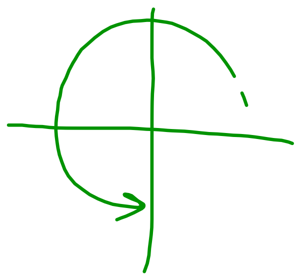
$= (0) + (0)$

$= 0$

R.S.

$\sin 3\pi$

$= 0$



Test $\frac{\pi}{2}$

L.S.

$\sin \frac{\pi}{2} + \sin 2\left(\frac{\pi}{2}\right)$

$= 1 + 0$

$= 1$

R.S.

$\sin 3\left(\frac{\pi}{2}\right)$

$= \sin \frac{3\pi}{2}$

$= -1$

L.S. $\neq$ R.S. $\therefore$ $\sin x + \sin 2x \neq \sin 3x$

Example 3

Prove that $\cos\left(\frac{\pi}{2}+x\right)=-\sin x$.

L.S.

$= \cos\left(\frac{\pi}{2}+x\right)$

$= \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x$

$= (0)\cos x - (1)\sin x$

$= -\sin x$

Example 4

Prove that $\frac{\cos(x-y)}{\cos(x+y)} = \frac{1+\tan x \tan y}{1-\tan x \tan y}$.

R.S.

$$= \frac{1+\tan x \tan y}{1-\tan x \tan y}$$

$$= \frac{1+\frac{\sin x \sin y}{\cos x \cos y}}{1-\frac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\frac{\cos x \cos y}{\cos x \cos y}+\frac{\sin x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y}-\frac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\frac{\cos x \cos y + \sin x \sin y}{\cancel{\cos x \cos y}}}{\frac{\cos x \cos y - \sin x \sin y}{\cancel{\cos x \cos y}}}$$

$$= \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y - \sin x \sin y}$$

$$= \frac{\cos(x-y)}{\cos(x+y)}$$

Example 5

Prove that $\tan 2x - 2\tan 2x \sin^2 x = \sin 2x$.

L.S.

$= \tan 2x - 2\tan 2x \sin^2 x$

$= \tan 2x(1 - 2\sin^2 x)$

$= \tan 2x(\cos 2x)$

$= \frac{\sin 2x}{\cancel{\cos 2x}}(\cancel{\cos 2x})$

L.S. = R.S. $\therefore \tan 2x - 2\tan 2x \sin^2 x = \sin 2x$

Consolidation

Hints

1. Create opportunities for items to "cancel".

2. When you see tan, think sin/cos

3. When you see csc, sec, or cot, think sin, cos, or tan.

4. Always look to see if an expression can be factored.

    * Look for differences of squares.

$(\sin^2\theta - \cos^2\theta)$

$(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)$

$(x^2 - 16)$

$(x+4)(x-4)$

Consolidation

Practice

Pg. 417

5, 9, 10, 11