Learning Goal: I will explore the intersection of two planes.

Minds On: How are the planes related?

Action: Intersection of Two Planes

Consolidation: Practice

How Many Ways?

In what ways can two planes interact?

1. Parallel (never meet)
2. They meet (intersect in a line)
3. Coincident (intersect in a plane)

Given Cartesian equations, how would you know which is the case?

1. If normals are scalar multiples, D values don't have same relationship.
2. Normals are <u>not</u> scalar multiples.
3. Equations are scalar multiples.

9.3 Intersection of Two Planes

Minds On: Consider the solution to the system of equations $x - y + z = 4$ and $x - y + z = 5$. Discuss how these planes might be related to each other.

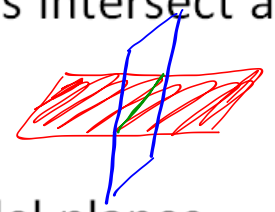
$\vec{n}_1 = (1,-1,1)$
$\vec{n}_2 = (1,-1,1)$ $\therefore \Pi_1 \parallel \Pi_2$

From Π_1 and Π_2, $4 \neq 5$ $\therefore$ noncoincident

$\Pi_1 \parallel \Pi_2$ and they never meet

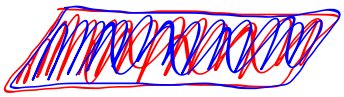
As with the intersection of lines, there are 3 cases for the intersection of two planes:

Case 1: Two planes intersect along a line



Case 2: Two parallel planes

Case 3: Two coincident planes



Solutions for a system of equations representing two planes

The system of equations corresponding to the intersection of two planes will have either zero solutions (parallel) or an infinite number of solutions (a line or planes are coincident). It is not possible for two planes to intersect at a single point.

Example 1: Determine the solution to the following system of equations:

① $x + 2y - 3z = -1$ ② $4x + 8y - 12z = -4$

$\vec{n}_1 = (1, 2, -3)$ $\vec{n}_2 = (4, 8, -12)$

$\vec{n}_2 = 4\vec{n}_1 \quad \therefore \Pi_1 \parallel \Pi_2$

From Π_2 and Π_1 $-4 = 4(-1)$

$\therefore \Pi_1$ and Π_2 are coincident.

Let $y = s$, $z = t$

From ① $x + 2s - 3t = -1$

$$\boxed{\begin{aligned} x &= -1 - 2s + 3t \\ y &= s \\ z &= t \end{aligned}}$$

$\therefore \vec{r} = (-1, 0, 0) + s(-2, 1, 0) + t(3, 0, 1)$

$s, t \in \mathbb{R}$

Intersection of Two Planes and their Normals

If the planes π_1 and π_2 have $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ as their respective normals, we know the following:

1. If $\overrightarrow{n_1} = k\overrightarrow{n_2}$ for some scalar, k, the planes are coincident or they are parallel and non-coincident. If they are coincident, there are an infinite number of points of intersection. If they are parallel and non-coincident, there are no points of intersection.

2. If $\overrightarrow{n_1} \neq k\overrightarrow{n_2}$, the two planes intersect in a line. This also results in an infinite number of points of intersection.

Example 2: Determine the solution to the following system of equations:

$x - y + z = 3$ $\qquad$ $2x + 2y - 2z = 3$

$\vec{n}_1 = (1, -1, 1)$ $\qquad$ $\vec{n}_2 = (2, 2, -2)$

Π_1 and Π_2 intersect in a line

① $x - y + z = 3$
② $2x + 2y - 2z = 3$

① $x - y + z = 3$
③ $0x + 4y - 4z = -3$ $\quad -2 \times ① + ②$

Let $z = t$

From ③ $4y - 4t = -3$

$4y = -3 + 4t$

$y = -\frac{3}{4} + t$

From ①

$$x - \left(-\frac{3}{4} + t\right) + t = 3$$

$$x + \frac{3}{4} \cancel{- t} \cancel{+ t} = 3$$

$$x = 3 - \frac{3}{4}$$

$$x = \frac{12}{4} - \frac{3}{4}$$

$$x = \frac{9}{4}$$

$\therefore$ the line where Π_1 and Π_2 intersect has equation $\vec{r} = \left(\frac{9}{4}, -\frac{3}{4}, 0\right) + t(0, 1, 1)$

Example 3: Determine the solution to the following system of equations:

① $2x - y + 3z = -2$ ② $x - 3z = 1$

$\vec{n}_1 = (2, -1, 3)$ $\vec{n}_2 = (1, 0, -3)$

Π_1 and Π_2 intersect in a line.

From ②, let $z = t$

$x - 3t = 1$

$x = 1 + 3t$

From ①

$2(1+3t) - y + 3t = -2$

$2 + 6t - y + 3t = -2$

$y = 4 + 9t$

$\therefore$ Π_1 and Π_2 intersect at the line with equation $\vec{r} = (1, 4, 0) + t(3, 9, 1)$

Example 4: Determine an equation of a line that passes through the point P(5, -2, 3) and is parallel to the line of intersection of the planes π_1: x + 2y – z = 6, Π_2: $y + 2z = 1$.

1. Find intersection of Π_1 and Π_2.
   ↳ the direction vector of line of intersection is our direction vector.
2. Write equation with given point and direction vector from step 1.

From Π_2, let $z = t$

$$y + 2t = 1$$
$$y = 1 - 2t$$

From Π_1, $x + 2(1-2t) - t = 6$

$$x + 2 - 4t - t = 6$$
$$x = 4 + 5t$$

direction vector for line of intersection is $(5,-2,1)$

$\therefore$ our line is

$$\vec{r} = (5,-2,3) + t(5,-2,1)$$