2.5 The Derivatives of Composite Functions

Example 1: If $f(x) = \sqrt{x}$ and $g(x) = x + 5$, find each of the following values:

a) $f(g(4))$ b) $g(f(4))$ c) $f(g(x))$ d) $g(f(x))$

The chain rule states how to compute the derivative of the composite function h(x) = f(g(x)) in terms of the derivatives of f and g.

The Chain Rule

If f and g are functions that have derivatives, then the composite function h(x) = f(g(x)) has a derivative given by h'(x) = f'(g(x))g'(x).

In Leibniz notation

If y is a function of u and u is a function of x (so that y is a composite function), then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

Example 2: Differentiate $h(x) = (x^2 + x)^{3/2}$

Example 3: If $y = u^3 - 2u + 1$, where $u = 2\sqrt{x}$, find $\frac{dy}{dx}$ at x = 4.

Example 4: An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide in the air can be modelled by the function $C(x) = \sqrt{0.5p^2 + 17}$, where C(p) is in parts per million and population p is expressed thousands. It is estimated that t years from now, the population of the community will be $p(t) = 3.1 + 0.1t^2$ thousand. At what rate will the carbon monoxide level be changing with respect to time three years from now?

Example 5: If $y = (x^2 - 5)^7$, find $\frac{dy}{dx}$.

Example 6: Differentiate $h(x) = (x^2 + 3)^4(4x - 5)^3$. Express your answer in a simplified factored form.

Example 7: Determine the derivative of $g(x) = \left(\frac{1+x^2}{1-x^2}\right)^{10}$