Learning Goal: I will be able to sketch the graphs of polynomial and rational functions.

Minds On: A Complete Sketch

Action: Algorithm for Sketching

Consolidation: Two Examples

Example 4: Determine any points of inflection on the graph of $f(x) = \frac{1}{x^2+3}$

Find $f''(x)$

↳ happens when $f''(x) = 0$ or DNE and graph goes from concave up to concave down or vice versa,

$f(x) = (x^2+3)^{-1}$

$f'(x) = -1(x^2+3)^{-2}(2x)$

$f'(x) = \frac{-2x}{(x^2+3)^2} = -2x(x^2+3)^{-2}$ ** Pg 205 #1,5

$f''(x) = -2(x^2+3)^{-2} + -2x(-2)(x^2+3)^{-3}(2x)$

$= \frac{-2(x^2+3)}{(x^2+3)^2(x^2+3)} + \frac{8x^2}{(x^2+3)^3}$

$= \frac{-2x^2-6+8x^2}{(x^2+3)^3}$

$= \frac{6x^2-6}{(x^2+3)^3}$

POI happen when $f''(x) = 0$ or is undefined and concavity changes.

- $f''(x)$ is always defined
- find when $f''(x) = 0$

$$\frac{6x^2-6}{(x^2+3)^3}=0$$

$6x^2-6=0$

$x^2=1$

$x=\pm 1$

Test $f''(x)$ on intervals $x<-1$, $-1<x<1$, $x>1$

* $(x^2+3)^3$ is always >0

	$x<-1$	$x=-1$	$-1<x<1$	$x=1$	$x>1$
$f''(x)$	+	0	−	0	+
graph	∪	POI	∩	POI	∪

when $x=+1$, $f(x)=\frac{1}{4}$

when $x=-1$, $f(x)=\frac{1}{4}$

$\therefore$ POI when $(x,y)=(-1,\frac{1}{4})$ & $(1,\frac{1}{4})$

***I'll leave this up for the first few minutes, check your solution and make sure you agree.**

Consolidation

Big Picture Stuff

If f'(x) is positive then f(x) ↑

If f'(x) is negative then f(x) ↓

If f'(x) is zero then f(x) is flat

If f'(x) is increasing then $f''(x) > 0$ and f(x) is concave Up

If f'(x) is decreasing then $f''(x) < 0$ and f(x) is concave down

If f'(x) is neither increasing nor decreasing then $f''(x) = 0$ and f(x) has a POI * maybe!

*test f''(x) on either side if it switches sign ☺

Minds On

A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

Asymptotes
- → Horizontal
- → Vertical
- → Oblique

Points of Inflection

Maxes / Mins (Critical Points)

Zeroes (x-intercepts)

y-intercept

Holes & Cusps

Intervals of ↑ & ↓

Concave ↑ and ↓

How do we determine each?

Minds On

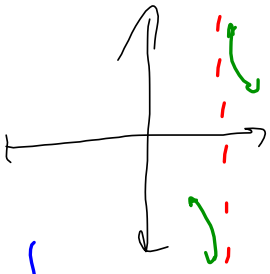
A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

How do we determine each?

Vertical Asymptotes



$f(x) = \frac{\#}{0}$

* test behaviour on either side

Minds On

A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

How do we determine each?

Horizontal Asymptotes

$\lim_{x \to \pm\infty} f(x) =$ some value

test behaviour above and below value by getting $f(x)$ with large $\oplus$ & $\ominus$ values

*weird factoring limit thing

Minds On

A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

How do we determine each?

Oblique Asymptotes

if $f(x) = \frac{p(x)}{q(x)}$ asymptote = $q(x)\overline{)p(x)}$

*happens if degree of p(x) is one higher than degree of q(x)

Minds On

A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

How do we determine each?

Points of Inflection

$f''(x) = 0$ or DNE

AND

$f''(x)$ changes from $\oplus$ to $\ominus$ or vice versa

Minds On

A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

How do we determine each?

Critical Points

$f'(x) = 0$

Max	Min	Neither
$f(x)$ ↑ then ↓	$f(x)$ ↓ then ↑	$f(x)$ ↑ then ↑ or ↓ then ↓

chart

OR

$f''(x) < 0$ $f''(x) > 0$

Minds On

A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

How do we determine each?

Intercepts

x-Intercepts

$f(x) = 0$

- factor
- quad. formula
- set = 0 and solve

y-Intercepts

$f(0) =$ ____

Minds On

A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

How do we determine each?

Intervals of Increase and Decrease

Increase

$f'(x) > 0$

Decrease

$f'(x) < 0$

test either side of critical points

OR

use common sense if max decreasing on either side

Minds On

A Proper Sketch

Groups - No Notes

What elements should be included in a complete sketch of a function?

How do we determine each?

Concavity

Up

$f''(x) > 0$

Down

$f''(x) < 0$

4.5 An Algorithm for Curve Sketching

As each of the following pieces of information are obtained, use them to build the sketch. When using this algorithm, keep two things in mind:

- You will not use all the steps in every situation. Use only the steps that are essential.
- You are familiar with the basic shapes of many functions. Use this knowledge when possible.

An Algorithm for Sketching the Graph of y = f(x)

1. Determine any discontinuities or limitations in the domain of f(x). For discontinuities, investigate the function's values on either side of the discontinuity. f(x)
2. Determine any vertical asymptotes.
3. Determine any intercepts. f(x)
4. Determine any critical numbers by finding where f'(x) = 0 or where f'(x) is undefined. f'(x)
5. Determine the intervals of increase/decrease, and then test critical points to see whether they are local maxima, local minima, or neither. May use f''(x) here f'(x) or f''(x)
6. Determine the behaviour of the function for large positive and large negative values of x. This will identify horizontal asymptotes, if they exist. Identify if the function's values approach the horizontal asymptote from above or below. $\lim_{x \to \pm\infty}$ f(x)
7. Determine f''(x) and test for points of inflection using the intervals of concavity. f''(x)
8. Complete the sketch using the above information.

Key Ideas

Use the function, f(x) to:
Determine the domain and any discontinuities; determine the intercepts, and find any asymptotes; and determine function behaviour relative to these asymptotes.

Use the first derivative, f'(x) to:
Find the critical numbers, determine where the function is increasing and where it is decreasing, identify any local maxima or minima.

Use the second derivative, f''(x) to:
Determine where the graph is concave up and where it is concave down, find any points of inflection, and the second derivative can also be used to identify local maxima and minima.

Consolidation

Sketch It

$f(x) = -3x^3 - 2x^2 + 5x$

Analyze f(x)

1. Asymptotes → No vertical } Polynomial
   → No horizontal

2. y-intercept → f(0) = 0 y int = (0,0)

3. x-intercepts → $f(x) = x(-3x^2 - 2x + 5)$

$= x(-3x^2 + 3x - 5x + 5)$

$= x(-3x(x-1) - 5(x-1))$

$= x(-3x-5)(x-1)$

$x(-3x-5)(x-1) = 0$

when $x = 0, 1, -\frac{5}{3}$

x-int = $(0,0), (1,0), (-\frac{5}{3}, 0)$

Analyze $f'(x)$

$f'(x) = -9x^2 - 4x + 5$

1. Critical Points ($f'(x) = 0$)

$-9x^2 - 4x + 5 = 0$

*factor or quadratic formula

$-9x^2 - 9x + 5x + 5 = 0$

$-9x(x+1) + 5(x+1) = 0$

$(-9x + 5)(x+1) = 0$

$f'(x) = 0$ when $x = \frac{5}{9}, -1$

$f\left(\frac{5}{9}\right) \doteq 1.65$ $f(-1) = -4$

or $f(0.56)$

critical points $(0.56, 1.65)$ and $(-1, -4)$

*will use $f''(x)$ to determine if these are maxes or mins

Analyze $f''(x)$

$f''(x) = -18x - 4$

1. Test critical points

$f''(0.56) = -14.08$ | $f''(-1) = 14$

∴ max (concave down) ∴ min (concave up)

∴ function is decreasing when $x < -1$,
increasing when $-1 < x < \frac{5}{9}$ and
decreasing when $x > \frac{5}{9}$

2. Look for POI + concavity

$f''(x) = 0$

$-18x - 4 = 0$

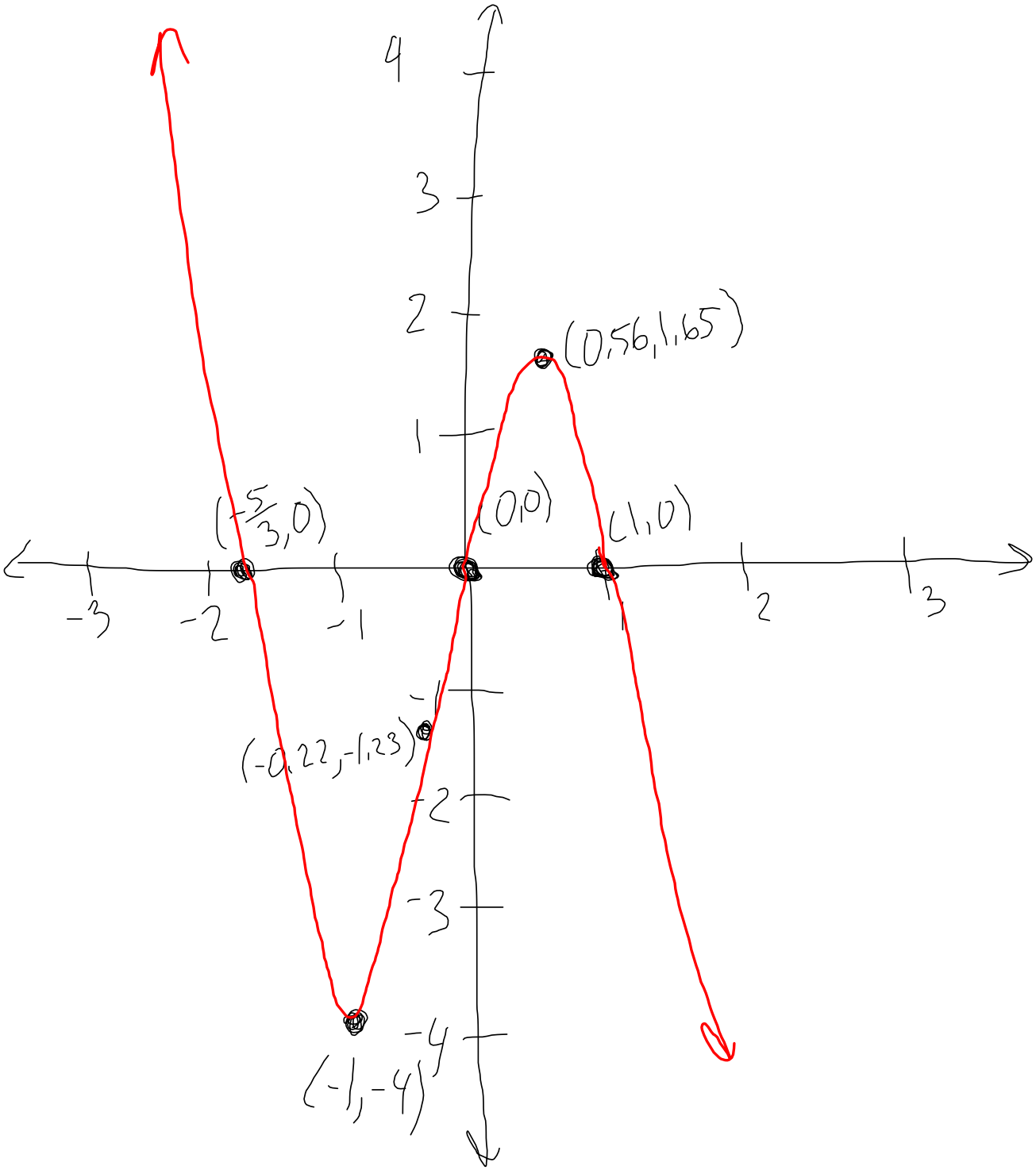
$-18x = 4$

$x = -\frac{2}{9}$

	$x < -\frac{2}{9}$	$x = -\frac{2}{9}$	$x > -\frac{2}{9}$
$f''(x)$	+	0	−
graph	concave up	POI	concave down

$f\left(-\frac{2}{9}\right) \doteq f(-0.22) \doteq -1.23$

POI at $(-0.22, -1.23)$


4
3
2
1
-1
-2
-3
-4
1
2
3
(0,56, 1,65)
(-5/3, 0)
(0,0)
(1,0)
(-0,22, -1,23)
(-1, -4)

Consolidation

Sketch It

$$f(x) = \frac{x - 4}{x^2 - x - 2}$$

Analyze f(x)

$$f(x) = \frac{x-4}{(x-2)(x+1)}$$

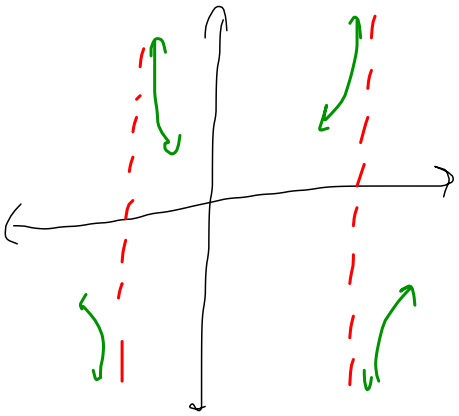
y-intercept: (0, 2)

x-intercept: (4, 0)

asymptotes: $x = -1$, $x = 2$

Test behaviour

	$x \to -1^-$ (-1.001)	$x \to -1^+$ (-0.999)	$x \to 2^-$ (1.999)	$x \to 2^+$ (2.001)
$(x-4)$	−	−	−	−
$(x-2)$	−	−	−	+
$(x+1)$	−	+	+	+
$f(x)$	−	+	+	−
graph →	$-\infty$	$+\infty$	$+\infty$	$-\infty$



Horizontal Asymptotes

$$f(x) = \frac{x-4}{x^2-x-2}$$

$$= \frac{x\left(1-\frac{4}{x}\right)}{x^2\left(1-\frac{1}{x}-\frac{2}{x^2}\right)}$$

$$\lim_{x\to+\infty} f(x) = \frac{\lim_{x\to+\infty}\left(1-\frac{4}{x}\right)}{\lim_{x\to+\infty} x\left(1-\frac{1}{x}-\frac{2}{x^2}\right)}$$

$$= \lim_{x\to+\infty} \frac{1}{x}$$

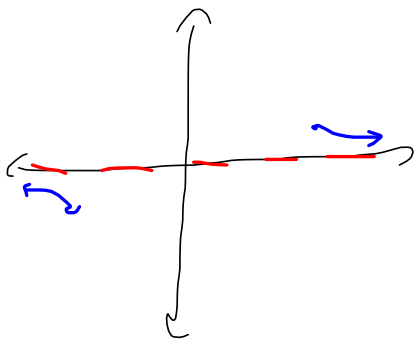
$$= 0 \quad \therefore \text{horizontal asymptote } y=0$$

$f(1000) = 0.000997$

as $x\to+\infty$, $f(x)$ approaches $y=0$ from above

$f(-1000) = -0.0010$

as $x\to-\infty$, $f(x)$ approaches $y=0$ from below



Analyze $f'(x)$

$f(x) = (x-4)(x^2-x-2)^{-1}$

$f'(x) = (1)(x^2-x-2)^{-1} + (x-4)(-1)(x^2-x-2)^{-2}(2x-1)$

$= \frac{1}{(x^2-x-2)} - \frac{(x-4)(2x-1)}{(x^2-x-2)}$

$= \frac{1}{(x^2-x-2)} - \frac{(2x^2-9x+4)}{(x^2-x-2)}$

$= \frac{(x^2-x-2)-(2x^2-9x+4)}{(x^2-x-2)^2}$

$= \frac{-x^2+8x-6}{(x^2-x-2)^2}$

$f'(x)=0$ when $-x^2+8x-6=0$ *can't factor!

$-(x^2-8x+6)=0$

$x = \frac{-8 \pm \sqrt{8^2-4(-1)(-6)}}{2(-1)}$

$x = \frac{-8+\sqrt{40}}{-2}$ or $\frac{-8-\sqrt{40}}{-2}$

$= 0.8$ or 7.2

$f(0.8) = 1.5$ $f(7.2) = 0.1$

$(0.8, 1.5)$ $(7.2, 0.1)$

Analyze $f(x)$ on either side of each or use second derivative test.

Let's look @ f''(x)

- $f'(x) = (-x^2+8x-6)(x^2-x-6)^{-2}$

$$f''(x) = (-2x+8)(x^2-x-6)^{-2} + (-x^2+8x-6)(-2)(x^2-x-6)^{-3}(2x-1)$$

$$= \frac{-2x+8}{(x^2-x-6)^2} + \frac{-2(2x-1)(-x^2+8x-6)}{(x^2-x-6)^3}$$

$$= \frac{(-2x+8)(x^2-x-6) - 2(2x-1)(-x^2+8x-6)}{(x^2-x-6)^3}$$

= gross!

The second derivative is overly complex. So we would test f'(x) on either side of each critical point to determine increasing/decreasing intervals and whether each point is a max or min.

We would not bother finding and POIs as it would be too difficult.