Learning Goal: I will be able to interpret the interactions between 3 planes.

Minds On: What are the possible options?

Action: Note: 9.4

Consolidation: Practice

3 Planes?

In what ways can 3 planes interact?

1. All parallel and non-coincident. ✗
2. All 3 coincident. ✓ plane
3. 2 Coincident, 1 parallel ✗
4. Two parallel, third intersects both ✗
5. Two coincident, third not ✓ line
6. Triangular prism formed ✗
7. All meet at a point ✓ point
8. All meet at a line ✓ line

9.4 The Intersection of Three Planes

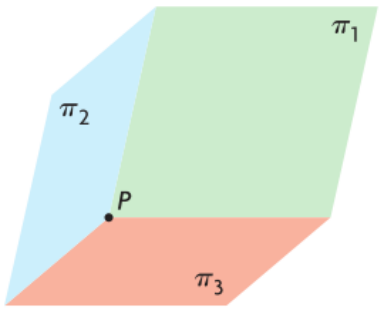
Minds On: In what ways might 3 planes interact? Use the table to classify your resonponses as Consistent Systems or Inconsistent Systems.

Consistent Systems	Inconsistent Systems

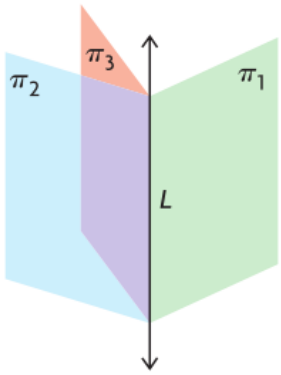
Intersection of 3 Planes

Consistent Systems

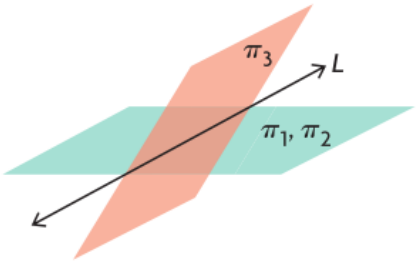
Case 1: P is the point of intersection of three planes.



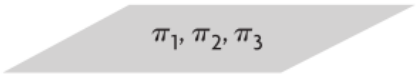
Case 2a: L is the line of intersection of three planes.



Case 2b: L is the line of intersection of two coincident planes and a third plane not parallel to the coincident planes.



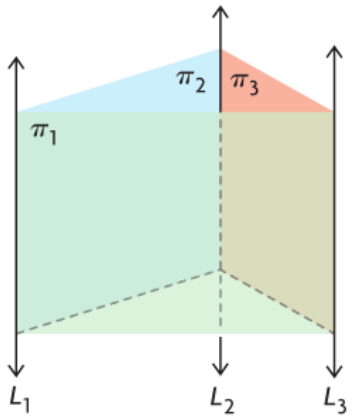
Case 3: The plane of intersection of three coincident planes is the plane itself.



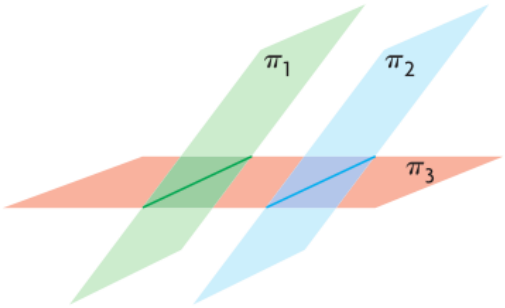
Intersection of 3 Planes

Inconsistent Systems

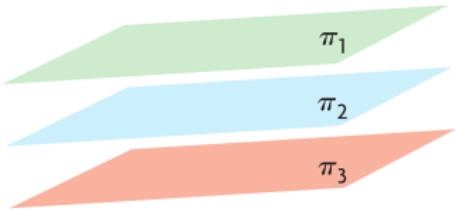
Case 1: A triangular prism is formed by three parallel lines.



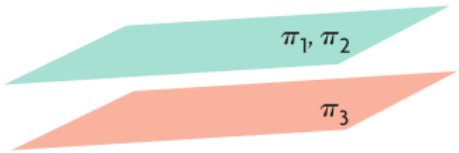
Case 2: Two parallel planes intersect a third plane.



Case 3: Three planes are parallel; none are coincident.



Case 4: Two planes are coincident; a third plane is parallel to and non-coincident with the first two planes.



Example 1: Determine the intersection of the three planes with the equations

$x - y + z = -2$, $2x - y - 2z = -9$, and $3x + y - z = -2$

① $x - y + z = -2$

② $2x - y - 2z = -9$

③ $3x + y - z = -2$

① $x - y + z = -2$

④ $0x + y - 4z = -5$ $-2 \times ① + ②$

⑤ $0x + 4y - 4z = 4$ $-3 \times ① + ③$

① $x - 4 + z = -2$

④ $0x + 4 - 4z = -5$

⑥ $0x + 0y + 12z = 24$ $-4 \times ④ + ⑤$

From ⑥, $\boxed{z = 2}$

From ④ $\boxed{y = 3}$

From ① $\boxed{x = -1}$

$\therefore$ the 3 planes intersect at the point $(-1, 3, 2)$

Example 2: Determine the solution to the following system of equations:

1) $2x - y + z = 1$
2) $3x - 5y + 4z = 3$
3) $3x + 2y - z = 0$

④ $x - \frac{1}{2}y + \frac{1}{2}z = \frac{1}{2}$ $\frac{1}{2} \times ①$

⑤ $0x - \frac{7}{2}y + \frac{5}{2}z = \frac{3}{2}$ $-3 \times ④ + ②$

⑥ $0x + \frac{7}{2}y - \frac{5}{2}z = -\frac{3}{2}$ $-3 \times ④ + ③$

④ $x - \frac{1}{2}y + \frac{1}{2}z = \frac{1}{2}$

⑦ $0x - 7y + 5z = 3$ $2 \times ⑤$

⑥ $0x + 7y - 5z = -3$ $2 \times ⑥$

(9) $x - \frac{1}{2}y + \frac{1}{2}z = \frac{1}{2}$

(7) $0x - 7y + 5z = 3$

$0x + 0y + 0z = 0$ (7) + (8)

$\therefore$ infinite number of solutions

Let $z = t$

From (7), $-7y + 5t = 3$

$$y = -\frac{3}{7} + \frac{5}{7}t$$

From (4), $x - \frac{1}{2}\left(-\frac{3}{7} + \frac{5}{7}t\right) + \frac{1}{2}t = \frac{1}{2}$

$x + \frac{3}{14} - \frac{5}{14}t + \frac{1}{2}t = \frac{1}{2}$

$$x = \frac{1}{2} - \frac{3}{14} + \frac{5}{14}t - \frac{1}{2}t$$

$$x = \frac{4}{14} - \frac{2}{14}t$$

$$\boxed{x = \frac{2}{7} - \frac{1}{7}t}$$

multiply direction numbers by 7

∴ the three planes meet when

$$x = \frac{2}{7} - t,\ y = -\frac{3}{7} + 5t,\ z = 7t$$

or

$$\vec{r} = \left(\frac{2}{7}, -\frac{3}{7}, 0\right) + t(-1, 5, 7)$$

Example 3: Determine the solution the following system of equations:

1. $2x + y + z = 1$ → not ∥, ∴ we have a line
2. $4x - y - z = 5$ } coincident
3. $8x - 2y - 2z = 10$ }

① $2x+y+z=1$

④ $0x-3y-3z=3$ $-2\times①+②$

⑤ $0x-6y-6z=6$ $-4\times①+③$

① $2x+y+z=1$

⑨ $0x-3y-3z=3$

$0x+0y+0z=0$ $-2\times④+⑤$

∴ infinite solutions

Let $z = t$

From ④, $-3y - 3t = 3$

$-3y = 3 + 3t$

$y = -1 - t$

From ①, $2x + (-1 - t) + (t) = 1$

$2x = 2$

$x = 1$

∴ solution is $x = 1, y = -1 - t, z = t$

or $\vec{r} = (1, -1, 0) + t(0, -1, 1)$

Example 4: Determine the solution the following systems of equations:

1. $x - y + z = 1$
2. $x + y + 2z = 2$
3. $x - 5y - z = 1$

} none ∥ ∴ non-coincident

① $x - y + z = 1$

④ $0x + 2y + z = 1$ $-1 \times ① + ②$

⑤ $0x - 4y - 2z = 0$ $-1 \times ① + ③$

① $x - y + z = 1$

④ $0x + 2y + z = 1$

⑥ $0x + 0y + 0z = 2$ $2 \times ④ + ⑤$

∴ no solution! they never meet

The planes make a triangular prism.

$\vec{n}_1 \times \vec{n}_2$

$\begin{matrix} -1 & 1 \\ 1 & 2 \\ 1 & 1 \\ -1 & 1 \end{matrix}$ $x = -3$, $y = -1$, $z = 2$ $(-3, -1, 2)$

$\vec{n}_2 \times \vec{n}_3$

$\begin{matrix} 1 & -5 \\ 2 & -1 \\ 1 & 1 \\ 1 & -5 \end{matrix}$ $x = 9$, $y = 3$, $z = -6$ $(9, 3, -6) = -3(-3, -1, 2)$

$\vec{n}_1 \times \vec{n}_3$

$\begin{matrix} -1 & -5 \\ 1 & -1 \\ 1 & 1 \\ -1 & -5 \end{matrix}$ $x = 6$, $y = 2$, $z = -4$ $(6, 2, -4) = -2(-3, -1, 2)$

$\therefore$ triangular prism

Example 5: Determine the solution the following systems of equations:

1. $x - y + 2z = -1$
2. $x - y + 2z = 3$
3. $x - 3y + z = 0$

① is parallel to ② and not coincident, ∴ no solution

Example 6: Solve the following system of equations:

1. $x + y + z = 5$
2. $x + y + z = 4$
3. $x + y + z = 5$

① and ③ are coincident

② is parallel to ① and ③ but not coincident, ∴ no solution.