Learning Goal: I will be able to use the second derivative to determine features of a graph.

Minds On: Distance-Time Graphs

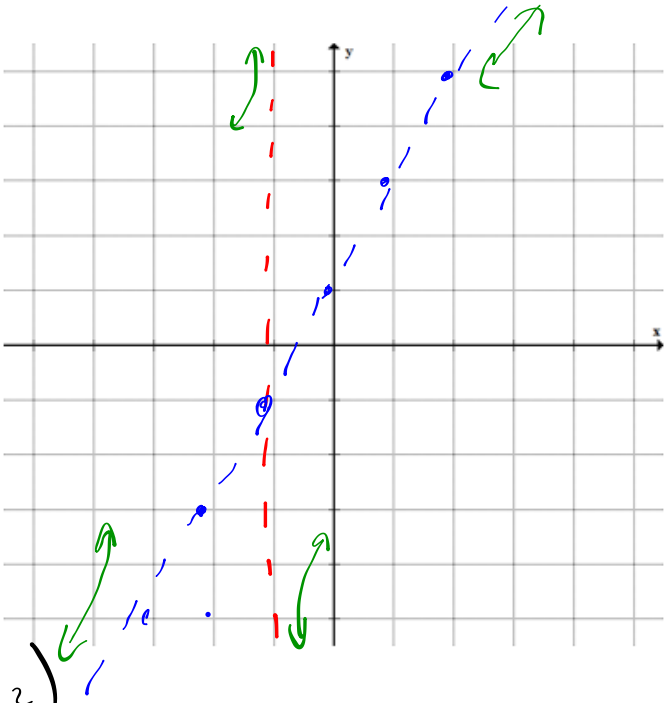
Action: Concavity and Points of Inflection

Consolidation: Exit Question

Example 6: Determine the equations of all asymptotes of the graph of $f(x) = \frac{2x^2+3x-1}{x+1}$.

vertical asymptotes: $x = -1$

	$x = -1.01$ $x \to -1^-$	$x = -0.99$ $x \to -1^+$
$f(x)$	$+$	$-$
$f(x) \to$	$+\infty$	$-\infty$



horizontal asymptotes

$$\lim_{x \to +\infty} \frac{2x^2+3x-1}{x+1}$$

$$= \lim_{x \to +\infty} \frac{2x^2\left(1+\frac{3}{2x}-\frac{1}{2x^2}\right)}{x\left(1+\frac{1}{x}\right)}$$

$$= \lim_{x \to +\infty} \frac{2x\left(1+\frac{3}{2x}-\frac{1}{2x^2}\right)}{\left(1+\frac{1}{x}\right)}$$

$= +\infty$ DNE $\therefore$ no horizontal asymptote

The degree of the numerator is 1 more than that of the denominator.

Oblique Asymptote

Divide

$$x+1 \overline{)\,2x^2+3x-1}$$

OR

$$\begin{array}{r|rrr} -1 & 2 & 3 & -1 \\ & & -2 & -1 \\ \hline & 2 & 1 & -2 \end{array}$$

← remainder (ignore)

$\therefore$ oblique asymptote is $2x+1$

Math Contests?

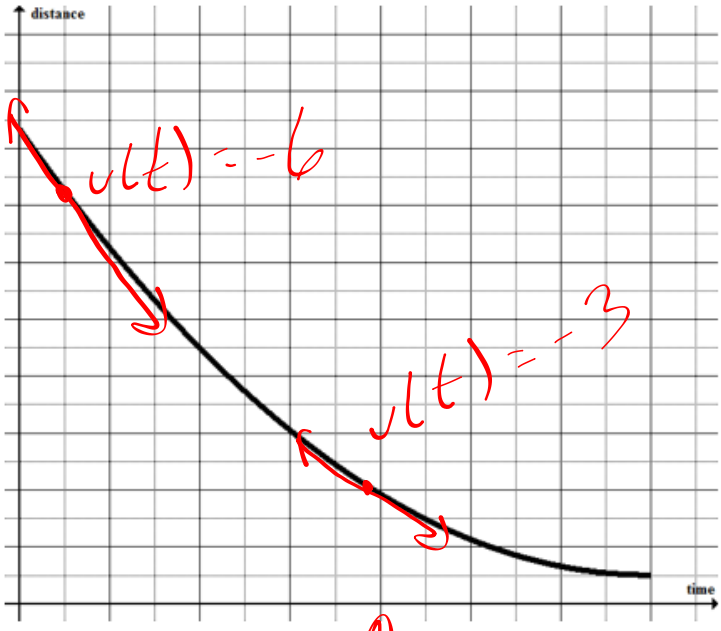
Euclid - April 11

Team - April 6 (unofficial)

Minds On

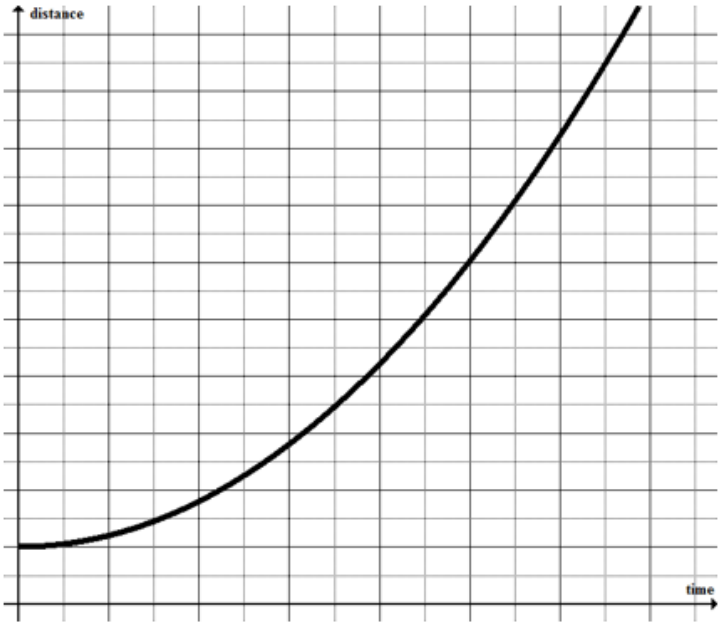
Distance-Time Graphs

The graph of a function f(x) is **concave up** on an interval if f'(x) is increasing on the interval. The graph of a function f(x) is **concave down** on an interval if f'(x) is decreasing on the interval.



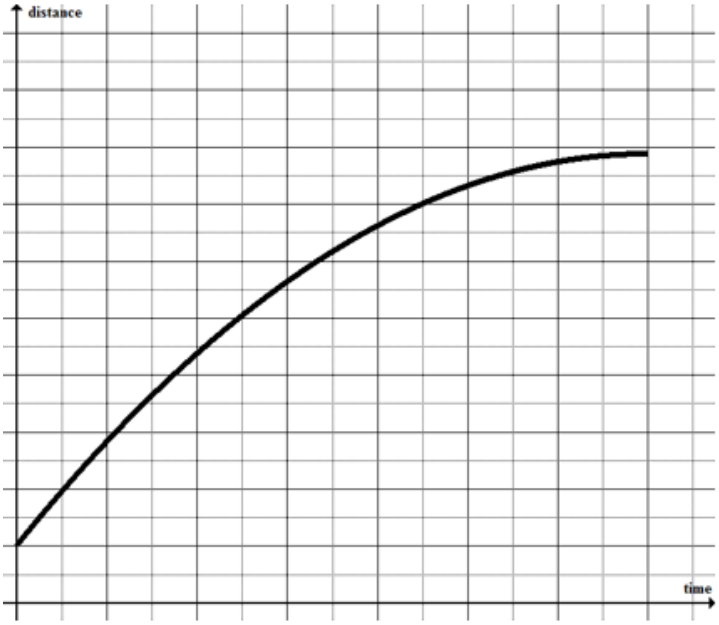
Velocity is _____

Acceleration is (+) _____



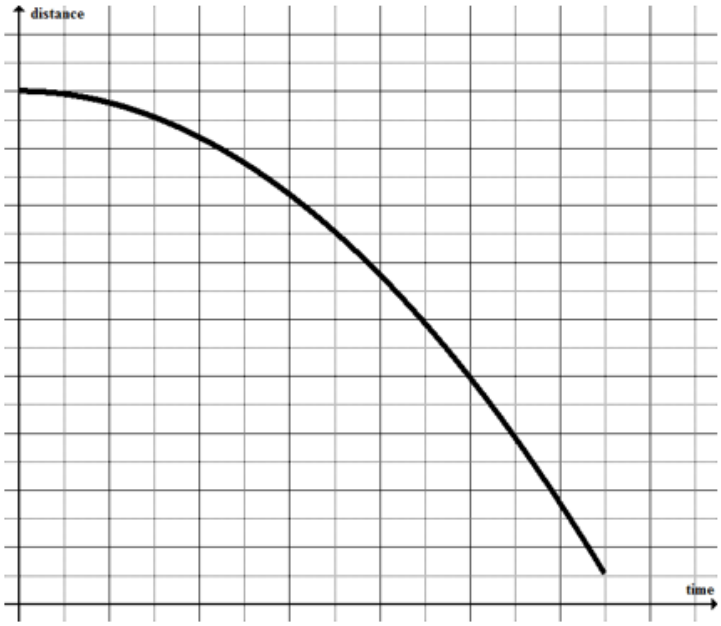
Velocity is _____

Acceleration is _____



Velocity is _____

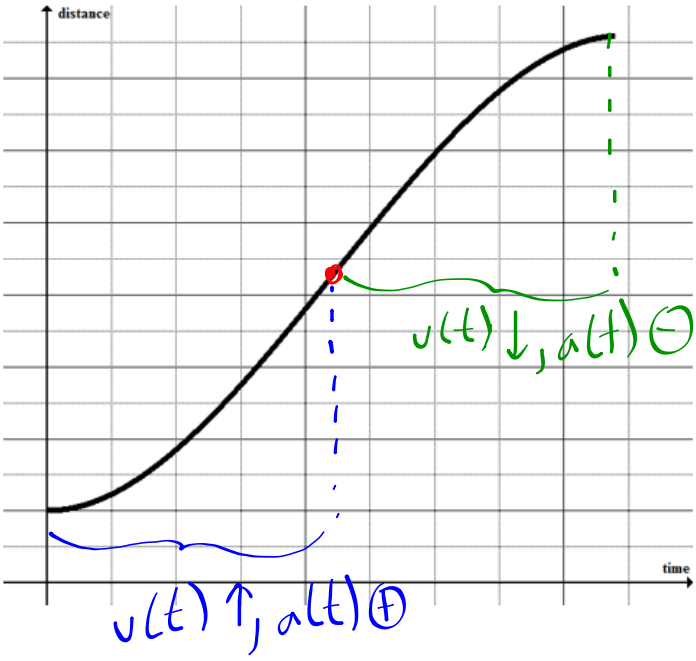
Acceleration is _____



Velocity is _____

Acceleration is _____

A **point of inflection** is a point on the graph of f(x) where the function changes from concave up to concave down, or vice versa.



Identify intervals over which the velocity is increasing and decreasing.

Describe the acceleration over each interval.

Action

Concavity and Points of Inflection

Test for Concavity: If f(x) is a differentiable function whose second derivative exists on an open interval l, then

- The graph of f(x) is concave up on l if f''(x) > 0 for all values of x in l
- The graph of f(x) is concave down on l if f''(x) < 0 for all values of x in l

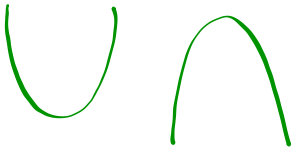
Test for Point of Inflection: If f(x) is a differentiable function whose second derivative exists on an interval containing c then

- The graph of f(x) has a point of inflection where the function changes from concave up to concave down, or vice versa.
- f''(c) = 0 or is undefined if (c, f(c)) is a point of inflection on the graph of f(x).

* if $f''(c) = 0$ we might have a point of inflection.

The second derivative test: Suppose that f(x) is a function for which f'(c) = 0, and the second derivative of f(x) exists on an interval containing c.

- If f''(c) > 0, then f(c) is a local minimum value.
- If f''(c) < 0, then f(c) is a local maximum value.
- If f''(c) = 0, then the test fails. Use the first derivative test.



Example 1: Sketch the graph of $y = x^3 - 3x^2 - 9x + 10$

1. Using f(x)

y-intercept = 10

2. Using f'(x)

$f'(x) = 3x^2 - 6x - 9$

$0 = 3x^2 - 6x - 9$

$0 = 3(x^2 - 2x - 3)$

$0 = 3(x-3)(x+1)$

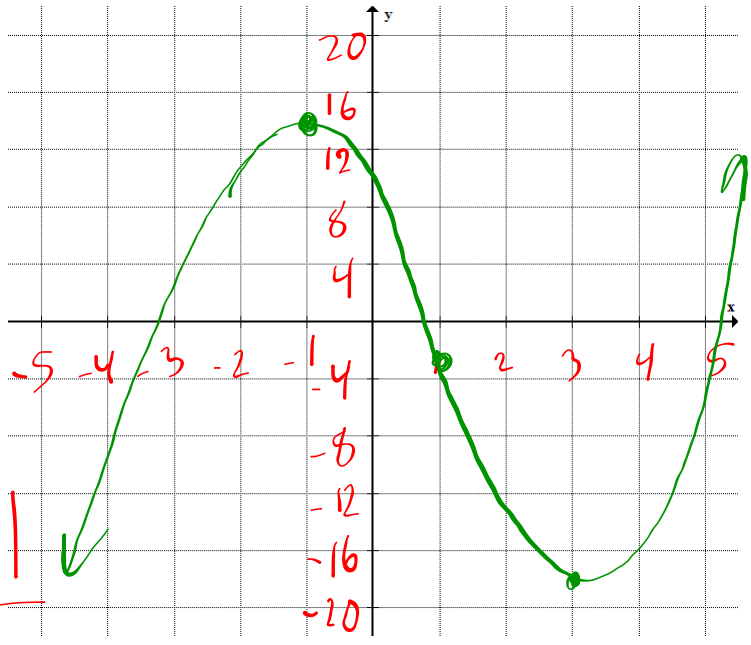
∴ critical points when $x = -1, 3$

3. Using f''(x)

$f''(x) = 6x - 6$

$0 = 6x - 6$

$x = 1$



	$x < 1$	$x > 1$
$6x-6$	−	+

second derivative changes from − to + ∴ we have a point of inflection when $x = 1$

Second Derivative Test

Test critical numbers

$f''(-1) = -12$ $\therefore$ max

$f''(3) = 12$ $\therefore$ min

Sub into $f(x)$ to get coordinates

$f(-1) = 15$

$f(3) = -17$

$f(1) = -1$

Example 2: Sketch the graph of $f(x) = x^4$

1. $f(x)$

y-int = 0
no asymptotes

2. $f'(x)$

$f'(x) = 4x^3$

$0 = 4x^3$

∴ critical point when $x = 0$

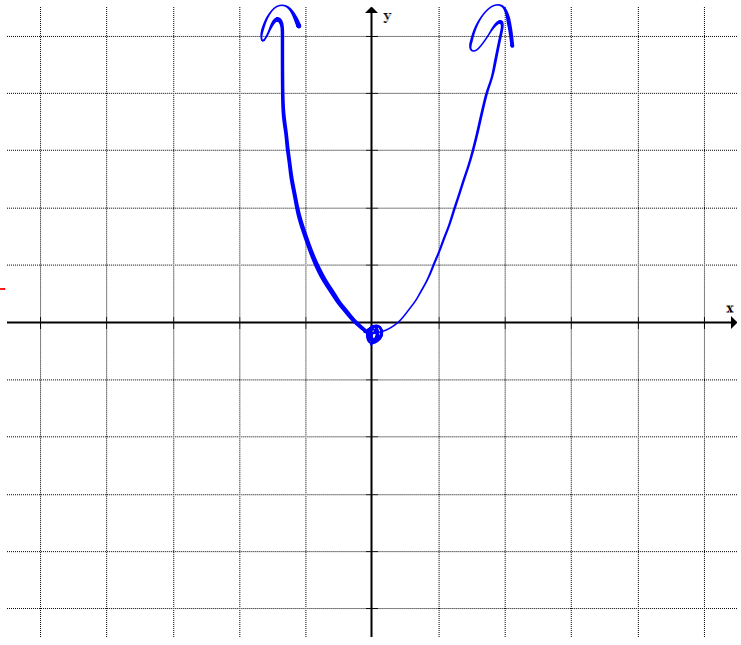
3. $f''(x)$

$f''(x) = 12x^2$

$0 = 12x^2$

$f''(x) = 0$ when $x = 0$

	$x < 0$	$x > 0$
$f''(x)$	+	+



no point of inflection because the function is concave up before and after $x = 0$, we have a min

Example 3: Sketch the graph of the function $f(x) = x^{1/3}$

1. $f(x)$

y-int = 0
no asymptotes

2. $f'(x)$

$f'(x) = \frac{1}{3}(x)^{-\frac{2}{3}}$

$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$

$0 = \frac{1}{3\sqrt[3]{x^2}}$ no maxes or mins

when $x=0$, $f'(x) = DNE$

3. $f''(x)$

$f''(x) = -\frac{2}{9}(x)^{-\frac{5}{3}}$

$f''(x) = \frac{-2}{9\sqrt[3]{x^5}}$

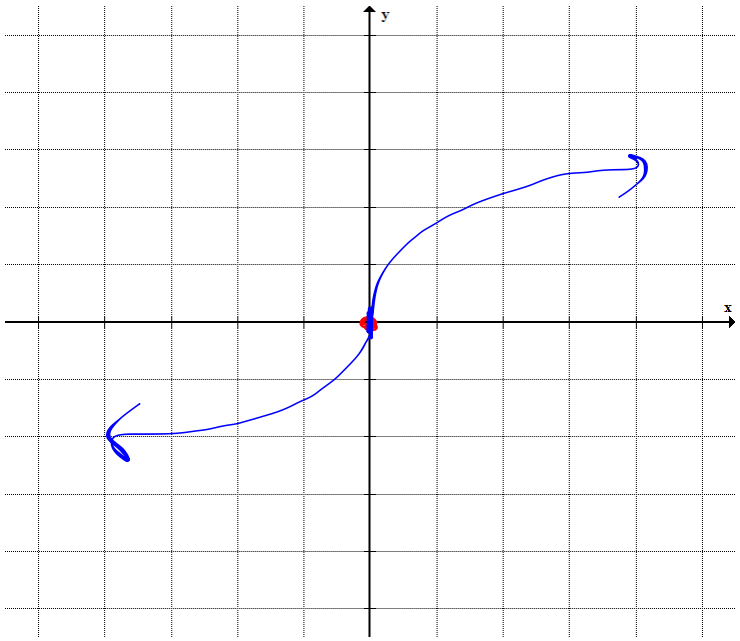
$f''(x)$ never $= 0$

when $x=0$, $f''(x) = DNE$

might have point of inflection when $x=0$

	$x<0$	$x>0$
$f''(x)$	+	−

∴ concavity changes from up to down and we have a point of inflection when $x=0$



Example 4: Determine any points of inflection on the graph of $f(x) = \frac{1}{x^2+3}$

$f(x) = (x^2+3)^{-1}$

$f'(x) = (-1)(x^2+3)^{-2}(2x)$

$= -2x(x^2+3)^{-2}$

$f''(x) = -2(x^2+3)^{-2} + -2x(-2)(x^2+3)^{-3}(2x)$

$= \frac{-2(x^2+3)}{(x^2+3)^3} + \frac{8x^2}{(x^2+3)^3}$

$= \frac{-2(x^2+3)+8x^2}{(x^2+3)^3}$

$= \frac{-2x^2-6+8x^2}{(x^2+3)^3}$

$= \frac{6x^2-6}{(x^2+3)^3}$

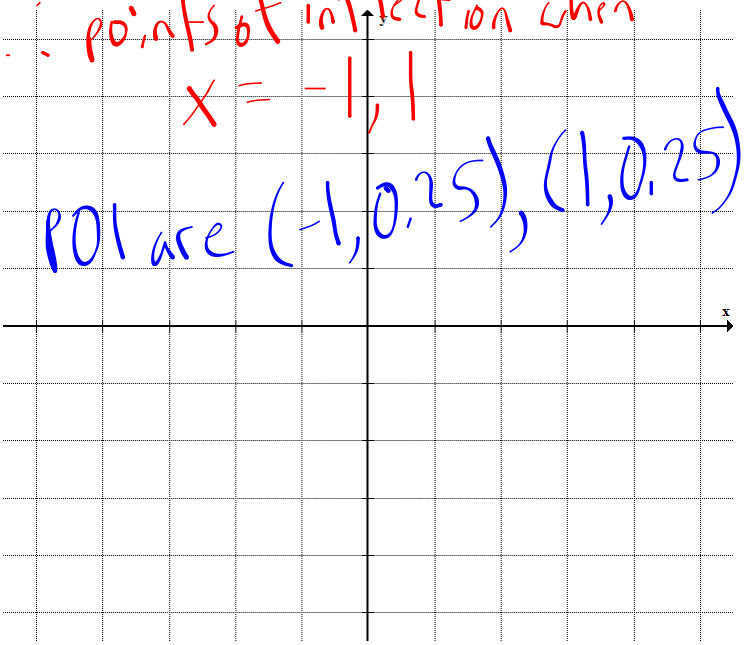
$0 = \frac{6x^2-6}{(x^2+3)^3}$

$0 = 6x^2-6$

$x^2 = 1$

$x = \pm 1$

	$x<-1$	$-1<x<1$	$x>1$
$6x^2-6$	+	−	+
$(x^2+3)^3$	+	+	+
$f''(x)$	+	−	+
concavity	up	down	up

∴ points of inflection when $x = -1, 1$

POI are $(-1, 0.25), (1, 0.25)$